

Asymptotic geometry of discrete interlaced patterns: Part I

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ABSTRACT. A discrete Gelfand-Tsetlin pattern is a configuration of particles in \mathbb{Z}^2 . The particles are arranged in a finite number of consecutive rows, numbered from the bottom. There is one particle on the first row, two particles on the second row, three particles on the third row, etc, and particles on adjacent rows satisfy an interlacing constraint.

We consider the uniform probability measure on the set of all discrete Gelfand-Tsetlin patterns of a fixed size where the particles on the top row are in deterministic positions. This measure arises naturally as an equivalent description of the uniform probability measure on the set of all tilings of certain polygons with lozenges. We prove a determinantal structure, and calculate the correlation kernel.

We consider the asymptotic behaviour of the system as the size increases under the assumption that the empirical distribution of the deterministic particles on the top row converges weakly. We consider the asymptotic ‘shape’ of such systems. We provide parameterisations of the asymptotic boundaries and investigate the local geometric properties of the resulting curves. We show that the boundary can be partitioned into natural sections which are determined by the behaviour of the roots of a function related to the correlation kernel. This paper should be regarded as a companion piece to the paper, [4], in which we resolve some of the remaining issues. Both of these papers serve as background material for the papers, [5] and [6], in which we examine the edge asymptotic behaviour.

1. Introduction

1.1. Random lozenge tilings of the regular hexagon. In this paper we consider the asymptotic shape of random tilings of ‘half-hexagons’. We define the systems in the next section, and note that they have a determinantal structure in section 1.4. Our motivation for studying such systems is to consider the local asymptotic boundary behaviour. Of course, in order to do this, one must first define and characterise a natural boundary. In section 1.5, under some natural asymptotic assumptions, we note that the asymptotic behaviour of the correlation kernel of the systems is amenable to steepest descent techniques. Essentially, the correlation kernel is expressed as a double contour integral (see equation (4)), and steepest descent techniques suggest that the asymptotic behaviour of the expression is determined by the behaviour of the roots of a certain analytic function (see equation (6)). In this paper, we define the edge, solely, by considering the behaviour

of the roots of this function. The edge is a natural boundary on which universal asymptotic behaviour is expected. This expectation is confirmed, using steepest descent techniques, in the paper [5]. In this paper, we also characterise other natural parts of the boundary. We continue this analysis in the paper [4], where we find previously unknown parts of the boundary. Finally, we find novel edge asymptotic behaviour in the paper [6].

In this section we introduce the systems of ‘half-hexagons’ by considering the simpler case of random tilings of a regular hexagon. The left hand side of figure 1 depicts a regular hexagon with sides of length $m \geq 1$. The middle depicts three different types of *lozenges* (polygons with angles $\frac{\pi}{3}$ and $\frac{2\pi}{3}$) with sides of length 1. We label these as types A, B, and C as shown. A complete covering of the interior of the hexagon with these lozenges is called a *tiling*. An example tiling, when $m = 4$, is given on the right of figure 1.

Note, in the example, lozenges of type A are adjacent to the left-most and right-most corners, lozenges of type B are adjacent to the bottom-left and top-right corners, and lozenges of type C are adjacent to the top-left and bottom-right corners. Given such a corner behaviour, consider a particular lozenge adjacent to a corner. The *connected component of this lozenge* is defined as the area covered by the set of all adjacent lozenges of the same type. Also, the *frozen region* is defined as the union of these six connected components. The remaining tiles form the so-called *disordered region*, and the boundary between the frozen and disordered regions is called the *frozen boundary*. The frozen boundary of the example tiling is shown on the left of figure 2.

Impose the uniform probability measure on the set of all possible tilings of a regular hexagon with sides of length m . Then, it is natural to consider the asymptotic behaviour of the system as $m \rightarrow \infty$. Some interesting results were obtained in [2]: The probability of observing the above corner behaviour converges to 1 as $m \rightarrow \infty$. For this reason we define:

DEFINITION 1.1. *A tiling of the regular hexagon is referred to as typical if and only if lozenges of type A are adjacent to the left-most and right-most corners, lozenges of type B are adjacent to the bottom-left and top-right corners, and lozenges of type C are adjacent to the top-left and bottom-right corners.*

Also, rescaling so that the sides of the hexagon are of length 1 for all m , the frozen boundary of typical random tilings converges to the inscribed circle of the rescaled hexagon as $m \rightarrow \infty$. This asymptotic shape is called the *Arctic circle*, and is shown on the right of figure 2. See [2] for more precise statements.

The goal of our work was to study the fluctuations of the frozen boundary around the asymptotic limit. We wanted to show that universal edge asymptotic behaviour holds. More exactly, we wanted to show that, when appropriately rescaled, the fluctuations of the frozen boundary converge to the *Airy process*. More generally, we wished to study the analogous question when tiling certain ‘half-hexagons’, which we define in the next section. Convergence to the 1-dimensional Airy process, in the case of the regular hexagon, was established by Baik *et al*, [1]. The analogous question for the frozen boundary of the Aztec diamond was settled by Johansson, [9]. Convergence to the 2-dimensional Airy process, in the case of the regular hexagon, was recently and independently established

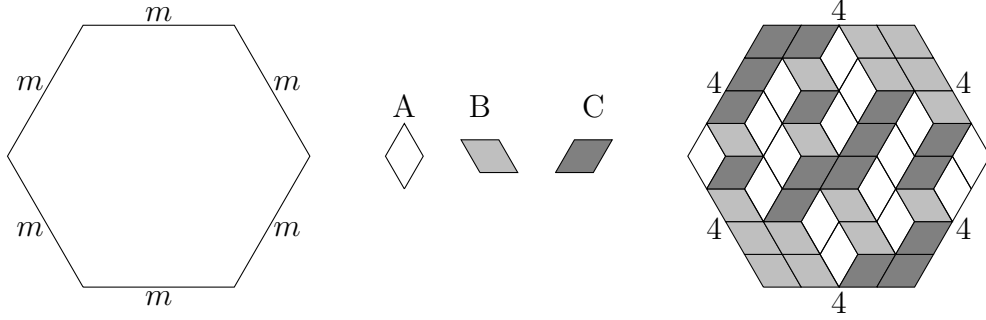


FIGURE 1. Left: A regular hexagon with sides of length $m \geq 1$.
 Middle: Three types of lozenges with sides of length 1.
 Right: An example tiling when $m = 4$.



FIGURE 2. Left: The frozen boundary of the example tiling of figure 1.
 Right: The Arctic circle, i.e., the asymptotic shape of the frozen boundary of a ‘typical random tiling’ as $m \rightarrow \infty$.

by Petrov, [19]. The Airy process has also been observed asymptotically at the edge of the spectra of various ensembles of random matrices (see, for example, Mehta, [16]). It was first observed in a study by Prähofer and Spohn, [20], of certain random growth models. See also Johansson, [8], which considers a different random growth model. It has also appeared asymptotically in other, seeming unrelated, systems.

The key to the asymptotic analysis in this and the accompanying papers is the investigation of a saddle-point function involving a known probability measure. This type of saddle point problem occurs in several contexts. A recent work which is closely related to ours is that of Hachem *et al.*, [7], which studies the asymptotic behaviour of the edge of the spectrum of large complex correlated Wishart matrices.

1.2. Random lozenge tilings of the ‘half-hexagon’ and equivalent interlaced particle configurations. We again begin by considering tilings of the regular hexagon. For simplicity we now restrict to the set of all typical tilings, defined in definition 1.1. It is not hard to see that any tiling is uniquely determined by the locations of the lozenges of type A, i.e., the *vertical lozenges*. Then, placing particles in the center of each vertical

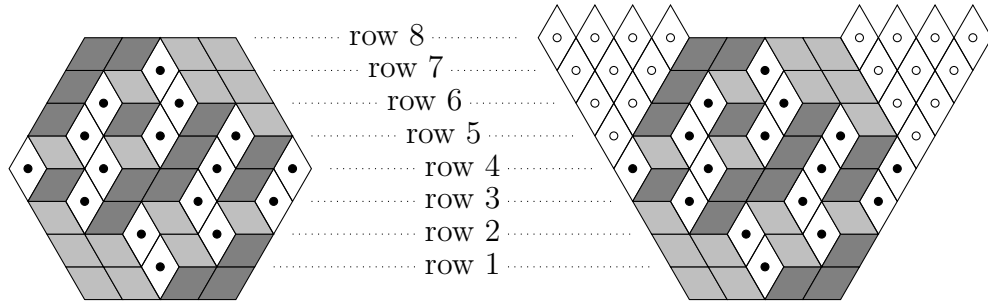


FIGURE 3. Left: Equivalent interlaced particle configuration of the example tiling of figure 1.

Right: Equivalent interlaced particle configuration with added deterministic lozenges/particles. The unfilled circles represent the deterministic particles.

lozenge, we see that the uniform measure on the set of all typical tilings is equivalent to the uniform measure on the set of all configurations of particles which satisfy:

- The particles are in the interior of the hexagon.
- The particles lie on $2m - 1$ rows, which we label from the bottom to the top.
- Adjacent rows are a distance of $\frac{\sqrt{3}}{2}$ apart.
- There are r distinct particles on row r when $r \in \{1, \dots, m\}$.
- There are $2m - r$ distinct particles on row r when $r \in \{m, \dots, 2m - 1\}$.
- Particles are in integer positions on the even rows.
- Particles are in half-integer positions (i.e., $\mathbb{Z} + \frac{1}{2}$) on the odd rows.
- Particles on adjacent rows *interlace*: When $r \in \{1, \dots, m - 1\}$, there is exactly one particle on row r ‘between’ each neighbouring pair of particles on row $r + 1$. Also, when $r \in \{m + 1, \dots, 2m - 1\}$, there is exactly one particle on row r ‘between’ each neighbouring pair of particles on row $r - 1$.

The particle configuration which is equivalent to the given example tiling is shown on the left of figure 3.

A further equivalent measure is obtained by adding deterministic lozenges/particles to each configuration in a particular way, as was done in Nordenstam, [18], and Petrov, [19]. This is demonstrated on the right of figure 3: We trivially add two densely packed blocks of lozenges/particles to the upper left and upper right sides of the hexagon. These deterministic lozenges/particles are independent of the tiling. Also, specifying the positions of the deterministic lozenges/particles on the top row, the interlacing constraint induces the positions of the deterministic lozenges/particles on the lower rows. Therefore the uniform measure on the set of all typical tilings is equivalent to the uniform measure on the set of all configurations of particles which satisfy:

- The particles lie on $2m$ rows, which we label from the bottom to the top.
- Adjacent rows are a distance of $\frac{\sqrt{3}}{2}$ apart.
- There are r distinct particles on row r for all $r \in \{1, \dots, 2m\}$.
- Particles are in integer positions on the even rows.

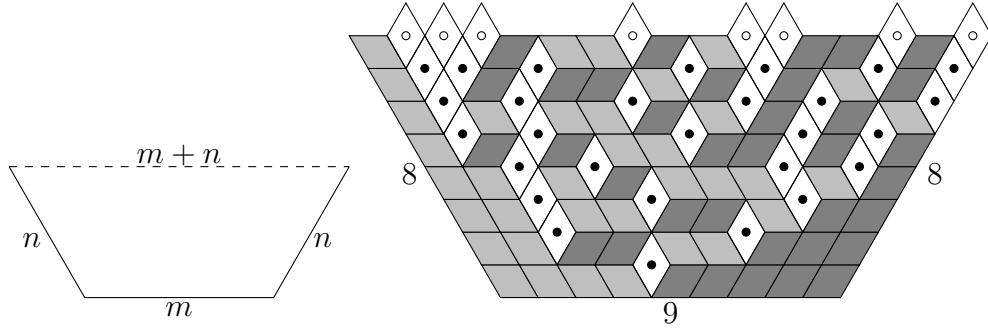


FIGURE 4. Left: A ‘half-hexagon’ with sides of length $n \geq 1$ and $m \geq 1$. The dotted line representing the upper boundary is considered to be ‘open’. Right: An example tiling and its equivalent interlaced particle configuration when $n = 8$ and $m = 9$. The unfilled circles represent the deterministic lozenges/particles.

- Particles are in half-integer positions (i.e., $\mathbb{Z} + \frac{1}{2}$) on the odd rows.
- The particles on row $2m$ are in the positions $\{1, \dots, m\} \cup \{2m+1, \dots, 3m\}$.
- Particles on adjacent rows *interlace*: For all $r \in \{1, \dots, 2m-1\}$, there is exactly one particle on row r ‘between’ each neighbouring pair of particles on row $r+1$.

A natural generalisation is to allow the lozenges/particles on the top row to be in arbitrary positions. More specifically, we consider the set of all lozenge tilings of the ‘half-hexagon’ with sides of length $n \geq 1$ and $m \geq 1$, as shown on the left of figure 4. We fix n vertical lozenges/particles in arbitrary deterministic integer positions on the upper boundary, and consider the uniform measure on the set of all possible tilings with this top row. An example of such a tiling, and its equivalent interlaced particle configuration, is shown on the right of figure 4.

Rescaling the sides of the ‘half-hexagon’ by $\frac{1}{n}$, we consider the asymptotic behaviour of the system as $n, m \rightarrow \infty$ under the assumption that $\frac{m}{n}$ converges to a positive constant and the empirical distribution of the deterministic lozenges/particles on the top row converges weakly. We consider the asymptotic ‘shape’ of such systems. Petrov, [19], studied the special case where the particles on the top row are contained in a finite number of densely packed blocks, and the empirical distribution converges to the Lebesgue measure restricted to a finite number of closed disjoint intervals. By adding deterministic lozenges/particles as for the regular hexagon, this is equivalent to tiling those types of polygons shown on the left of figure 5. The results of this paper hold for any probability measure that can be obtained as the weak limit of the empirical distribution.

We end this section by comparing and contrasting our results to those of Kenyon *et al.*, [12] and [13]. The asymptotic frozen boundaries of the polygons of figure 5, for example, can be studied using the techniques of these papers. The boundaries are shown to be *algebraic*. As stated above, this paper and [4] studies the frozen boundaries of a natural generalisation of those polygons shown on the left of figure 5. The techniques of Kenyon *et al.* do not cover such models, and our techniques do not cover the polygons shown

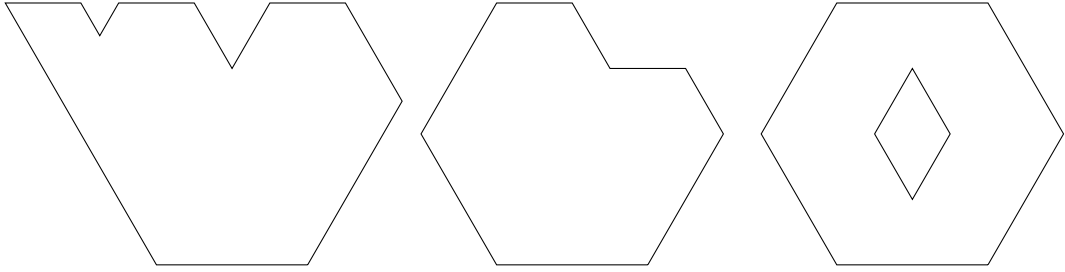


FIGURE 5. Polygons of various shapes that can be tiled with lozenges.
 Left: A ‘half-hexagon’ with V-shaped cuts on the top boundary.
 Center: A regular hexagon with a corner removed.
 Right: A regular hexagon with a diamond shaped hole in the center.

in the middle and on the right. In our case, the asymptotic frozen boundaries are not necessarily algebraic. We do, however, obtain parameterisations of the boundaries, and we perform a detailed analysis of their local geometric properties. Finally note, in [13], the asymptotic frozen boundary of the polygon in the middle is shown to be a cardioid. In [6], we consider a related situation, and study the asymptotic behaviour of particles in a neighbourhood of a cusp in the frozen boundary. These do not behave as a *Pearcey* point process, as previously expected. We obtain a novel point process, which we call the *Cusp Airy* process. This process can also appear for those polygons studied by Petrov, i.e., those shown on the left of figure 5. An example of such a polygon and cusp is given in section 2.5.4. The polygon and cusp in question can be seen in figure 12.

1.3. Determinantal random point processes. In this section we give a brief introduction to determinantal random point processes that suffices for our purposes. See Johansson, [10], for a more complete treatment.

Let Λ be a Polish space. Fix $N \in \mathbb{N} \cup \{\infty\}$ and $Y \subset \Lambda^N$, a space of configurations of N -particles of Λ . Denote each $y \in Y$ as $y = (y_1, \dots, y_N)$. Assume, for all $y \in Y$ and compact Borel sets $B \subset \Lambda$, that the number of particles from y contained in B is finite, i.e., $\#\{y_i \in B\} < \infty$. Let \mathcal{F} be the sigma-algebra generated by sets of the form $\{y \in Y : \#\{y_i \in B\} = m\}$ for all $m \leq N$ and Borel sets $B \subset \Lambda$. A probability space of the form $(Y, \mathcal{F}, \mathbb{P})$ is referred to as a *random point process*.

Given such a process, $m \leq N$, and $B \subset \Lambda^m$, define $N_B^m : Y \rightarrow \mathbb{N}$ by,

$$N_B^m(y) := \#\{(y_{i_1}, \dots, y_{i_m}) \in B : i_1 \neq \dots \neq i_m\},$$

for all $y \in Y$. In words, $N_B^m(y)$ is the number of distinct m -tuples of particles from y that are contained in B . Then define a measure on Λ^m by $B \mapsto \mathbb{E}[N_B^m]$ for all Borel subsets $B \subset \Lambda^m$. Assume that this is well-defined and finite whenever B is bounded. Then, given a reference measure λ on Λ , the density of the above measure with respect to λ^m , whenever it exists, is referred to as the m^{th} *correlation function*, ρ_m . That is,

$$\int_B \rho_m(x_1, \dots, x_m) d\lambda[x_1] \dots d\lambda[x_m] = \mathbb{E}[N_B^m],$$

for all Borel subsets $B \subset \Lambda^m$.

A random point process is called *determinantal* if all correlation functions exist and there exists a function $K : \Lambda^2 \rightarrow \mathbb{C}$ for which

$$\rho_m(x_1, \dots, x_m) = \det[K(x_i, x_j)]_{i,j=1}^m,$$

for all $x_1, \dots, x_m \in \Lambda$ and $m \leq N$. K is called the *correlation kernel* of the process.

1.4. The determinantal structure of discrete Gelfand-Tsetlin patterns. For the remainder of the paper we restrict to the study of the interlaced particle configurations that we introduced in section 1.2. Recall that adjacent rows are a distance of $\frac{\sqrt{3}}{2}$ apart, and that particles on adjacent rows alternate between integer and half-integer positions. Therefore, since it is more convenient to study configurations of particles in \mathbb{Z}^2 , we shift each row r vertically by $-(1 - \frac{\sqrt{3}}{2})(n - r)$, and horizontally by $\frac{1}{2}(n - r)$. This preserves the interlacing constraint, and the resulting configuration is called a *Gelfand-Tsetlin pattern*, which we now define rigorously:

DEFINITION 1.2. *A discrete Gelfand-Tsetlin pattern of depth n is an n -tuple, denoted $(y^{(1)}, y^{(2)}, \dots, y^{(n)}) \in \mathbb{Z} \times \mathbb{Z}^2 \times \dots \times \mathbb{Z}^n$, which satisfies the interlacing constraint*

$$y_1^{(r+1)} \geq y_1^{(r)} > y_2^{(r+1)} \geq y_2^{(r)} > \dots \geq y_r^{(r)} > y_{r+1}^{(r+1)},$$

for all $r \in \{1, \dots, n-1\}$, denoted $y^{(r+1)} \succ y^{(r)}$. Equivalently this can be considered as an interlaced configuration of $\frac{1}{2}n(n+1)$ particles in $\mathbb{Z} \times \{1, \dots, n\}$ by placing a particle at position $(u, r) \in \mathbb{Z} \times \{1, \dots, n\}$ whenever u is an element of $y^{(r)}$.

A Gelfand-Tsetlin pattern of depth 4 is shown on the left of figure 6.

For each $n \geq 1$, fix $x^{(n)} \in \mathbb{Z}^n$ with $x_1^{(n)} > x_2^{(n)} > \dots > x_n^{(n)}$. Consider the uniform probability measure, ν_n , on the set of discrete Gelfand-Tsetlin patterns of depth n with the particles on row n in the deterministic positions defined by $x^{(n)}$:

$$\nu_n[(y^{(1)}, \dots, y^{(n)})] := \frac{1}{Z_n} \cdot \begin{cases} 1 & \text{; when } x^{(n)} = y^{(n)} \succ y^{(n-1)} \succ \dots \succ y^{(1)}, \\ 0 & \text{; otherwise,} \end{cases}$$

where $Z_n > 0$ is a normalisation constant. This measure, and the equivalent description of Gelfand-Tsetlin patterns given in definition 1.2, induces a random point process on interlaced configurations of particles in $\mathbb{Z} \times \{1, \dots, n\}$. This process is determinantal, a fact which follows from the equivalent description of the system as perfect matchings of hexagonal planar graphs (see, for example, Kenyon, [11]). This observation does not, however, provide a convenient expression for the correlation kernel of the process. In section 4.1, we use the Gelfand-Tsetlin description to find such an expression. We denote the correlation kernel by $K_n : (\mathbb{Z} \times \{1, \dots, n\})^2 \rightarrow \mathbb{C}$, a function of pairs of particle positions. Ignoring the deterministic particles on row n , interlacing implies that we need only consider those particle positions, $(u, r), (v, s) \in \mathbb{Z} \times \{1, \dots, n-1\}$, which satisfy $u \geq x_n^{(n)} + n - r$ and $v \geq x_n^{(n)} + n - s$. For all such $(u, r), (v, s)$, we show in section 4.1 that

$$(1) \quad K_n((u, r), (v, s)) = \tilde{K}_n((u, r), (v, s)) - \phi_{r,s}(u, v),$$

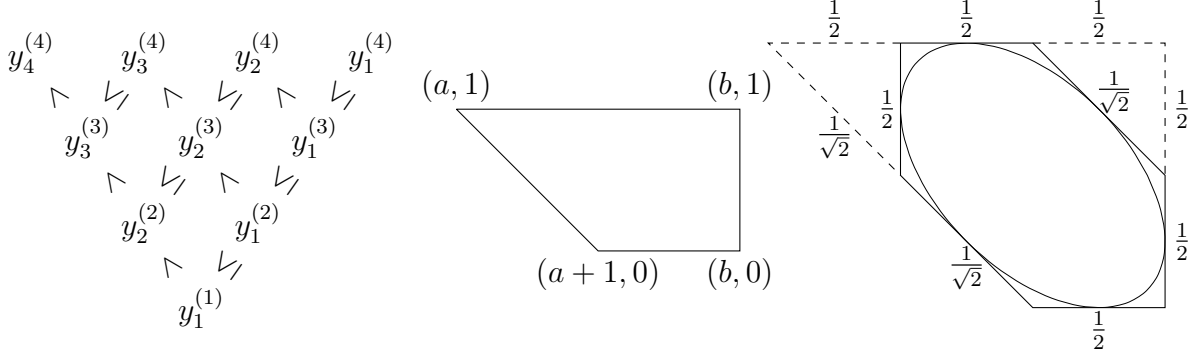


FIGURE 6. Left: A visualisation of a Gelfand-Tsetlin pattern of depth 4. Middle: $\{(\chi, \eta) \in \mathbb{R} \times [0, 1] : \chi \in [a + 1 - \eta, b]\}$, where a and b satisfy $b - a > 1$ (see hypothesis 1.1). Equations (2) and (3) imply that the bulk of the rescaled particles of the Gelfand-Tsetlin patterns lie asymptotically in this region as $n \rightarrow \infty$. Right: The shifted asymptotic shape of the rescaled regular hexagon. The areas enclosed by the dashed lines represent the added regions of deterministic lozenges/particles, as described in section 1.2.

where

$$\begin{aligned} & \tilde{K}_n((u, r), (v, s)) \\ &:= \frac{(n-s)!}{(n-r-1)!} \sum_{k=1}^n 1_{(x_j^{(n)} \geq u)} \sum_{l=v+s-n}^v \frac{\prod_{j=u+r-n+1}^{u-1} (x_k^{(n)} - j)}{\prod_{j=v+s-n, j \neq l}^v (l-j)} \prod_{i=1, i \neq k}^n \left(\frac{l - x_i^{(n)}}{x_k^{(n)} - x_i^{(n)}} \right), \end{aligned}$$

and

$$\phi_{r,s}(u, v) := 1_{(v \geq u)} \cdot \begin{cases} 0 & ; \text{ when } s \leq r, \\ 1 & ; \text{ when } s = r + 1, \\ \frac{1}{(s-r-1)!} \prod_{j=1}^{s-r-1} (v - u + s - r - j) & ; \text{ when } s > r + 1. \end{cases}$$

This is a generalisation of Defosseux, [3], and Metcalfe, [17], which consider a similar process on configurations in $\mathbb{R} \times \{1, \dots, n\}$. The kernel in [3] and [17] is recovered from the above kernel using asymptotic arguments. The above kernel was also independently obtained by Petrov, [19]. Our proof, based on the methods used in [3] and [17], is more elementary than that of Petrov. We highlight the differences at the beginning of section 4.1.

1.5. Motivation and statement of main results. In this paper we study the asymptotic behaviour of the determinantal system introduced in the previous section as $n \rightarrow \infty$, under the assumption that the (rescaled) empirical distribution of the deterministic particles on row n converges weakly to a measure with compact support:

HYPOTHESIS 1.1. *Let μ be a probability measure on \mathbb{R} with compact support, $\text{Supp}(\mu) \subset [a, b]$. Assume that $b - a > 1$, $\{a, b\} \subset \text{Supp}(\mu)$, and*

$$\frac{1}{n} \sum_{i=1}^n \delta_{x_i^{(n)}/n} \rightarrow \mu$$

as $n \rightarrow \infty$, in the sense of weak convergence of measures.

REMARK 1.1. *Recalling that $\{x_1^{(n)}, x_2^{(n)}, \dots, x_n^{(n)}\} \subset \mathbb{Z}$, it trivially follows that measures, μ , which satisfy hypothesis 1.1 are absolutely continuous with respect to Lebesgue measure, λ . Moreover, the density of μ takes values in $[0, 1]$, and so $\mu \leq \lambda$. Finally, $\text{Supp}(\mu)$ and $\text{Supp}(\lambda - \mu)$ contain no isolated singletons.*

For each n , we now rescale the vertical and horizontal positions of the particles of the Gelfand-Tsetlin pattern of depth n by $\frac{1}{n}$. Recall that $\{x_1^{(n)}, x_2^{(n)}, \dots, x_n^{(n)}\}$ are the (deterministic) particles on the top row of the Gelfand-Tsetlin pattern of depth n . Note, hypothesis 1.1 easily gives,

$$(2) \quad \#\left\{x \in \left\{\frac{x_1^{(n)}}{n}, \frac{x_2^{(n)}}{n}, \dots, \frac{x_n^{(n)}}{n}\right\} : x \in [a, b]\right\} = n + o(n),$$

for all n sufficiently large. Thus, at most $o(n)$ of the rescaled particles on the top row are not contained in the interval $[a, b]$ for all n sufficiently large. Next, fix any $\eta \in [0, 1]$, and any $\{m_n\}_{n \geq 1} \subset \mathbb{Z}$ with $m_n \in \{1, 2, \dots, n-1\}$ and $\frac{m_n}{n} = \eta + o(1)$ for all n sufficiently large. Also, let $\{y_1^{(m_n)}, y_2^{(m_n)}, \dots, y_{m_n}^{(m_n)}\}$ denote the (random) particles on row m_n of the Gelfand-Tsetlin pattern of depth n . Equation (2) and the interlacing constraint then give,

$$(3) \quad \#\left\{y \in \left\{\frac{y_1^{(m_n)}}{n}, \frac{y_2^{(m_n)}}{n}, \dots, \frac{y_{m_n}^{(m_n)}}{n}\right\} : y \in [a + 1 - \eta, b]\right\} = m_n + o(n),$$

for all n sufficiently large. Thus, at most $o(n)$ of the rescaled particles on row m_n are not contained in the interval $[a + 1 - \eta, b]$ for all n sufficiently large. Thus, since η is any value in $[0, 1]$, the bulk of the rescaled particles of the Gelfand-Tsetlin patterns lie asymptotically in $\{(\chi, \eta) \in \mathbb{R} \times [0, 1] : \chi \in [a + 1 - \eta, b]\}$ as $n \rightarrow \infty$. This geometric subset of \mathbb{R}^2 is shown in the middle of figure 6. The example of the regular hexagon is shown on the right of figure 6. This figure is obtained from figure 2 by performing the shift described at the beginning of section 1.4. We recover this figure in section 2.5.3, using our techniques.

The local asymptotic behaviour of particles near a point, (χ, η) , in the shape in the middle of figure 6, can be examined by considering the asymptotic behaviour of $K_n((u_n, r_n), (v_n, s_n))$ as $n \rightarrow \infty$, where $\{(u_n, r_n)\}_{n \geq 1}$ and $\{(v_n, s_n)\}_{n \geq 1}$ are sequences in \mathbb{Z}^2 which satisfy:

HYPOTHESIS 1.2. *Fix (χ, η) in the shape in middle of figure 6, i.e., $(\chi, \eta) \in [a, b] \times [0, 1]$ with $b \geq \chi \geq \chi + \eta - 1 \geq a$. Assume that $\frac{1}{n}(u_n, r_n) \rightarrow (\chi, \eta)$ and $\frac{1}{n}(v_n, s_n) \rightarrow (\chi, \eta)$ as $n \rightarrow \infty$.*

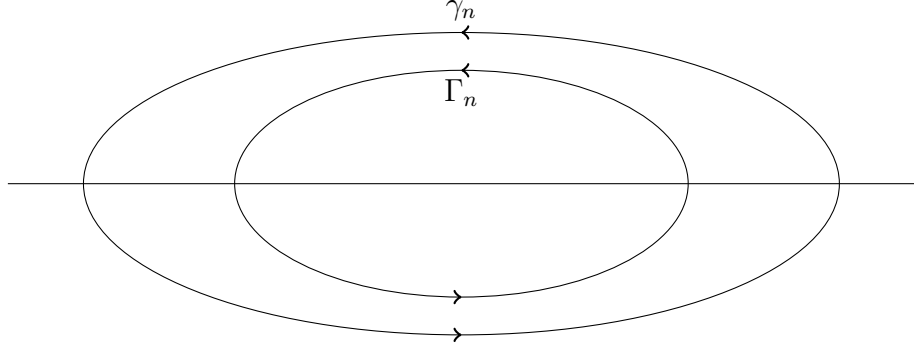


FIGURE 7. Γ_n contains $\{\frac{1}{n}x_j : x_j \geq u_n\}$ and none of $\{\frac{1}{n}x_j : x_j \leq u_n + r_n - n\}$. γ_n contains Γ_n and $\{\frac{1}{n}v_n, \frac{1}{n}(v_n - 1), \dots, \frac{1}{n}(v_n + s_n - n)\}$. Both contours are oriented counter-clockwise.

The asymptotic behaviour of $K_n((u_n, r_n), (v_n, s_n))$ can be examined using steepest descent techniques. To see this, first note that equation (1) and the Residue Theorem give,

$$(4) \quad K_n((u_n, r_n), (v_n, s_n)) = \left(\frac{(n - s_n)!}{(n - r_n - 1)!} \frac{n^{n-r_n-1}}{n^{n-s_n}} \right) J_n - \phi_{r_n, s_n}(u_n, v_n),$$

where, dropping the superscript from $x^{(n)}$,

$$(5) \quad J_n := \frac{1}{(2\pi i)^2} \int_{\gamma_n} dw \int_{\Gamma_n} dz \frac{\prod_{j=u_n+r_n-n+1}^{u_n-1} (z - \frac{j}{n})}{\prod_{j=v_n+s_n-n}^{v_n} (w - \frac{j}{n})} \frac{1}{w - z} \prod_{i=1}^n \left(\frac{w - \frac{x_i}{n}}{z - \frac{x_i}{n}} \right),$$

for all $n \in \mathbb{N}$, where γ_n and Γ_n are any counter-clockwise closed contours that satisfy the requirements of figure 7. Also note that the integrand can be written as

$$\frac{\exp(n f_n(w) - n \tilde{f}_n(z))}{w - z},$$

for all $w, z \in \mathbb{C} \setminus \mathbb{R}$, where

$$f_n(w) := \frac{1}{n} \sum_{i=1}^n \log \left(w - \frac{x_i}{n} \right) - \frac{1}{n} \sum_{j=v_n+s_n-n}^{v_n} \log \left(w - \frac{j}{n} \right),$$

$$\tilde{f}_n(z) := \frac{1}{n} \sum_{i=1}^n \log \left(z - \frac{x_i}{n} \right) - \frac{1}{n} \sum_{j=u_n+r_n-n+1}^{u_n-1} \log \left(z - \frac{j}{n} \right),$$

and \log denotes the principal logarithm. Finally, inspired by hypotheses 1.1 and 1.2 we define

$$f_{(\chi, \eta)}(w) := \int_a^b \log(w - x) \mu[dx] - \int_{\chi+\eta-1}^{\chi} \log(w - x) dx,$$

for all $w \in \mathbb{C} \setminus \mathbb{R}$. Note that $f_n(w) \rightarrow f_{(\chi, \eta)}(w)$ and $\tilde{f}_n(z) \rightarrow f_{(\chi, \eta)}(z)$ as $n \rightarrow \infty$ for all $w, z \in \mathbb{C} \setminus \mathbb{R}$.

The above expression for the integrand, and the asymptotic behaviour of the functions f_n and \tilde{f}_n in the exponent, suggest that the contour integral representation of the kernel is amenable to steepest descent techniques. More exactly, steepest descent analysis suggests that, as $n \rightarrow \infty$, the asymptotic behaviour of $K_n((u_n, r_n), (v_n, s_n))$ depends on the behaviour of the roots of $f'_{(\chi, \eta)}$:

$$(6) \quad f'_{(\chi, \eta)}(w) = \int_a^b \frac{\mu[dx]}{w - x} - \int_{\chi+\eta-1}^{\chi} \frac{dx}{w - x},$$

for all $w \in \mathbb{C} \setminus \mathbb{R}$. Note also that hypotheses 1.1 and 1.2 imply that $\mu[\chi, b] \geq 0$, $(\lambda - \mu)[\chi + \eta - 1, \chi] \geq 0$ and $\mu[a, \chi + \eta - 1] \geq 0$, where λ is Lebesgue measure. It is therefore natural to write,

$$(7) \quad f'_{(\chi, \eta)}(w) = \int_{\chi}^b \frac{\mu[dx]}{w - x} - \int_{\chi+\eta-1}^{\chi} \frac{(\lambda - \mu)[dx]}{w - x} + \int_a^{\chi+\eta-1} \frac{\mu[dx]}{w - x},$$

for all $w \in \mathbb{C} \setminus \mathbb{R}$. Therefore $f'_{(\chi, \eta)}$ extends analytically to

$$\mathbb{C} \setminus (\text{Supp}(\mu|_{[\chi, b]}) \cup \text{Supp}((\lambda - \mu)|_{[\chi+\eta-1, \chi]}) \cup \text{Supp}(\mu|_{[a, \chi+\eta-1]})).$$

The main result of section 3, theorem 3.1, characterises all possible behaviours of the roots of $f'_{(\chi, \eta)}$ in this domain.

REMARK 1.2. *For simplicity of notation, we omit the subscript from $f_{(\chi, \eta)}$ whenever confusion is impossible. Moreover, note that the first term on the right hand side of equation (6) is the function,*

$$w \mapsto \int_a^b \frac{\mu[dx]}{w - x},$$

for all $w \in \mathbb{C} \setminus \mathbb{R}$. This function extends analytically to $\mathbb{C} \setminus \text{Supp}(\mu)$, is conventionally called the Cauchy transform of μ , and is denoted by C .

We now use the behaviours of the roots observed in theorem 3.1 to divide the shape in the middle of figure 6 into regions in which different asymptotic behaviours can be expected. An important region is the following:

DEFINITION 1.3. *The liquid region, \mathcal{L} , is the set of all (χ, η) in the shape in middle of figure 6, i.e., $(\chi, \eta) \in [a, b] \times [0, 1]$ and $b \geq \chi \geq \chi + \eta - 1 \geq a$, for which f' has non-real roots.*

In section 2 we consider the geometric interpretation of this region. First note that non-real roots of f' occur in complex conjugate pairs. Theorem 3.1 then implies that $(\chi, \eta) \in \mathcal{L}$ if and only if f' has exactly 2 roots in $\mathbb{C} \setminus \mathbb{R}$, counting multiplicities. More exactly, there exists a unique $w_c \in \mathbb{H} := \{w \in \mathbb{C} : \text{Im}(w) > 0\}$ with $f'(w_c) = f'(\overline{w}_c) = 0$. This defines a map from \mathcal{L} to \mathbb{H} . In theorem 2.1 we show that this map is a homeomorphism (indeed, it is a diffeomorphism, as we shall see in section 4.2). Therefore \mathcal{L} is a non-empty,

open, connected set. Moreover, an explicit expression is obtained for the inverse of the homeomorphism, denoted by,

$$(\chi_{\mathcal{L}}(\cdot), \eta_{\mathcal{L}}(\cdot)) : \mathbb{H} \rightarrow \mathcal{L} \subset [a, b] \times [0, 1].$$

Steepest descent analysis and the above observations suggest, as $n \rightarrow \infty$, that universal bulk asymptotic behaviour should be observed whenever $(\chi, \eta) \in \mathcal{L}$: Fixing $(\chi, \eta) \in \mathcal{L}$, and choosing the parameters (u_n, r_n) and (v_n, s_n) of hypothesis 1.2 appropriately, it should be possible to show that $K_n((u_n, r_n), (v_n, s_n))$ converges to the *Sine* kernel as $n \rightarrow \infty$. Note that an analogous result was obtained in Metcalfe, [17], which considers similar processes on configurations of particles in $\mathbb{R} \times \{1, \dots, n\}$. In the current situation, Petrov, [19], confirmed universal bulk asymptotic behaviour when the measure, μ , of hypothesis 1.1 is given by the Lebesgue measure restricted to a finite number of closed disjoint intervals.

In section 2.2 we use the above homeomorphism to examine $\partial\mathcal{L}$, the boundary of \mathcal{L} . The main result of this section, lemma 2.3, defines a subset of $\partial\mathcal{L}$ for any measure, μ , of hypothesis 1.1. This is defined by showing that $(\chi_{\mathcal{L}}(\cdot), \eta_{\mathcal{L}}(\cdot)) : \mathbb{H} \rightarrow \mathcal{L}$ has a unique continuous extension to the open set $R \subset \mathbb{R}$ given by,

$$(8) \quad R := (\overline{(\mathbb{R} \setminus \text{Supp}(\mu)) \cup (\mathbb{R} \setminus \text{Supp}(\lambda - \mu))})^\circ.$$

Above we take the interior of the closure of the union. The extension is denoted by,

$$(\chi_{\mathcal{E}}(\cdot), \eta_{\mathcal{E}}(\cdot)) : R \rightarrow [a, b] \times [0, 1],$$

and an explicit expression is given. We conclude that $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) \in \partial\mathcal{L}$ for all $t \in R$, and $(\chi_{\mathcal{E}}(\cdot), \eta_{\mathcal{E}}(\cdot)) : R \rightarrow \partial\mathcal{L}$ is a smooth curve parameterised over R . Also, as we shall shortly see, this map is injective. Finally we define:

DEFINITION 1.4. *The edge, $\mathcal{E} \subset \partial\mathcal{L}$, is the image of the smooth curve $(\chi_{\mathcal{E}}(\cdot), \eta_{\mathcal{E}}(\cdot)) : R \rightarrow \partial\mathcal{L}$.*

Lemma 2.3, and the other results of section 2.2, give a complete description of $\partial\mathcal{L}$ only when the measure μ , of hypothesis 1.1, is restricted to an interesting sub-class of possible measures (see lemma 2.5). When μ is not restricted to the sub-class, these results only give a partial description. Figure 8, below, depicts those sections of $\partial\mathcal{L}$ obtained from these results for various examples of μ . The examples are examined in detail in section 2.5.

In section 2.3 we construct an alternative description of the edge, \mathcal{E} , in terms of the behaviour of the roots of f' . This is analogous to definition 1.3 for the liquid region, \mathcal{L} :

DEFINITION 1.5. *The edge, \mathcal{E} , is the disjoint union $\mathcal{E} := \mathcal{E}_\mu \cup \mathcal{E}_{\lambda-\mu} \cup \mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2$, where*

- \mathcal{E}_μ is the set of all (χ, η) in the shape in the middle of figure 6 for which f' has a repeated root in $\mathbb{R} \setminus [\chi + \eta - 1, \chi]$.
- $\mathcal{E}_{\lambda-\mu}$ is the set of all (χ, η) for which f' has a repeated root in $(\chi + \eta - 1, \chi)$.
- \mathcal{E}_0 is the set of all (χ, η) for which $\eta = 1$ and f' has a root at χ ($= \chi + \eta - 1$).
- \mathcal{E}_1 is the set of all (χ, η) for which $\eta < 1$ and f' has a root at χ .
- \mathcal{E}_2 is the set of all (χ, η) for which $\eta < 1$ and f' has a root at $\chi + \eta - 1$.

The fact that the above union is disjoint, and that \mathcal{L} and \mathcal{E} are disjoint, follows from corollary 3.2. To see that the definitions are equivalent, note that theorem 3.1 implies that f' has at most one real-valued repeated root. Then, starting with definition 1.5, we can define a map from \mathcal{E} to \mathbb{R} by mapping $(\chi, \eta) \in \mathcal{E}$ to the relevant root of f' , i.e.,

- to the unique real-valued repeated root whenever $(\chi, \eta) \in \mathcal{E}_\mu \cup \mathcal{E}_{\lambda-\mu}$.
- to the root χ whenever $(\chi, \eta) \in \mathcal{E}_0 \cup \mathcal{E}_1$.
- to the root $\chi + \eta - 1$ whenever $(\chi, \eta) \in \mathcal{E}_0 \cup \mathcal{E}_2$.

Theorem 2.3 implies that this bijectively maps \mathcal{E} to R , defined above. Moreover, the inverse of this map is the smooth curve of definition 1.4, i.e., $(\chi_{\mathcal{E}}(\cdot), \eta_{\mathcal{E}}(\cdot)) : R \rightarrow \partial\mathcal{L}$. Thus the curve is a smooth and bijective map from R to the edge, \mathcal{E} , of definition 1.5. Therefore the definitions are trivially equivalent.

We end section 2.3 with lemma 2.6, which further clarifies the equivalence of the above definitions. First recall that $\mu \leq \lambda$ (see remark 1.1), and note that $\mathbb{R} \setminus \text{Supp}(\mu)$ and $\mathbb{R} \setminus \text{Supp}(\lambda - \mu)$ are disjoint open sets. Equation (8) thus gives

$$R = (\mathbb{R} \setminus \text{Supp}(\mu)) \cup (\mathbb{R} \setminus \text{Supp}(\lambda - \mu)) \cup R_1 \cup R_2,$$

where

- R_1 is the set of all $t \in \mathbb{R}$ for which there exists an $\epsilon > 0$ with $(t, t+\epsilon) \subset \mathbb{R} \setminus \text{Supp}(\mu)$ and $(t-\epsilon, t) \subset \mathbb{R} \setminus \text{Supp}(\lambda - \mu)$.
- R_2 is the set of all $t \in \mathbb{R}$ for which there exists an $\epsilon > 0$ with $(t, t+\epsilon) \subset \mathbb{R} \setminus \text{Supp}(\lambda - \mu)$ and $(t-\epsilon, t) \subset \mathbb{R} \setminus \text{Supp}(\mu)$.

Intuitively, one can think of R_1 as the set of all points where the density of μ jumps from 1 (on the left) to 0 (on the right), and R_2 as the set of all points where it jumps from 0 to 1. Next, fix $t \in R$, and define,

$$(\chi, \eta) := (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) \quad \text{and} \quad f'_t := f'_{(\chi, \eta)},$$

where the last term is the function given in equation (7). The equivalence of definitions 1.4 and 1.5, discussed above, implies that t is a root of f'_t . Then, letting $C : \mathbb{C} \setminus \text{Supp}(\mu) \rightarrow \mathbb{C}$ be the Cauchy transform of μ (see remark 1.2), and letting $m_t \geq 1$ denote the multiplicity of t as a root of f'_t , lemma 2.6 implies that the following 9 cases exhaust all possibilities:

- (1) $(\chi, \eta) \in \mathcal{E}_\mu$, $t \in \mathbb{R} \setminus \text{Supp}(\mu)$ with $C(t) \neq 0$, and $m_t = 2$.
- (2) $(\chi, \eta) \in \mathcal{E}_\mu$, $t \in \mathbb{R} \setminus \text{Supp}(\mu)$ with $C(t) \neq 0$, and $m_t = 3$.
- (3) $(\chi, \eta) \in \mathcal{E}_{\lambda-\mu}$, $t \in \mathbb{R} \setminus \text{Supp}(\lambda - \mu)$, and $m_t = 2$.
- (4) $(\chi, \eta) \in \mathcal{E}_{\lambda-\mu}$, $t \in \mathbb{R} \setminus \text{Supp}(\lambda - \mu)$, and $m_t = 3$.
- (5) $(\chi, \eta) \in \mathcal{E}_0$, $t \in \mathbb{R} \setminus \text{Supp}(\mu)$ with $C(t) = 0$, and $m_t = 1$.
- (6) $(\chi, \eta) \in \mathcal{E}_1$, $t \in R_1$, and $m_t = 1$.
- (7) $(\chi, \eta) \in \mathcal{E}_1$, $t \in R_1$, and $m_t = 2$.
- (8) $(\chi, \eta) \in \mathcal{E}_2$, $t \in R_2$, and $m_t = 1$.
- (9) $(\chi, \eta) \in \mathcal{E}_2$, $t \in R_2$, and $m_t = 2$.

In section 2.4 we investigate the local geometric properties of the edge curve. First, fix $t \in R$, and define the (un-normalised) orthogonal vectors, $\mathbf{x} = \mathbf{x}(t)$ and $\mathbf{y} = \mathbf{y}(t)$, as

in lemma 2.7. Also define $(\chi, \eta) := (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$. A Taylor expansion of the edge curve then gives (see lemma 2.7),

$$(\chi_{\mathcal{E}}(s), \eta_{\mathcal{E}}(s)) - (\chi, \eta) = a(s)\mathbf{x} + b(s)\mathbf{y},$$

for all $s \in R$ sufficiently close to t , where

$$\begin{aligned} a(s) &= a_1(s-t) + a_2(s-t)^2 + O((s-t)^3), \\ b(s) &= b_1(s-t)^2 + b_2(s-t)^3 + O((s-t)^4), \end{aligned}$$

and $a_1 = a_1(t)$, $a_2 = a_2(t)$, $b_1 = b_1(t)$ and $b_2 = b_2(t)$ are known. Next, we investigate a_1, a_2, b_1 and b_2 for each of the exhaustive cases, (1-9) discussed above. In lemma 2.8 we show that:

- $a_1 \neq 0$ and $b_1 \neq 0$ in cases (1, 3, 5, 6, 8).
- $a_1 = b_1 = 0$, $a_2 \neq 0$ and $b_2 \neq 0$ in cases (2, 4, 7, 9).

The above Taylor expansion then implies that the edge curve behaves like a parabola in a neighbourhood of (χ, η) in cases (1, 3, 5, 6, 8), with tangent vector \mathbf{x} and normal vector \mathbf{y} . Also the edge curve behaves like an algebraic cusp of first order in a neighbourhood of (χ, η) in cases (2, 4, 7, 9), and the vector \mathbf{x} can be said to define the ‘orientation’ of the cusp. Also, since cases (1-9) are exhaustive, no other behaviour is possible.

In the paper, [5], we use steepest descent techniques to examine the edge asymptotic behaviour for cases (1-4). Recall that in case (1), $t \in \mathbb{R} \setminus \text{Supp}(\mu)$ is a root of f'_t of multiplicity 2, and the edge curve behaves locally like a parabola in a neighbourhood of $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$. In case (2), $t \in \mathbb{R} \setminus \text{Supp}(\mu)$ is a root of f'_t of multiplicity 3, and the edge curve behaves like an algebraic cusp of first order in a neighbourhood of $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$. In [5], we confirm universal edge asymptotic behaviour for these cases: As $n \rightarrow \infty$, choosing the parameters (u_n, r_n) and (v_n, s_n) of hypothesis 1.2 appropriately, $K_n((u_n, r_n), (v_n, s_n))$ converges to the *Airy* kernel in case (1), and the *Pearcey* kernel in case (2). We also show a similar result in cases (3) and (4), except now the asymptotic behaviour of the correlation kernel of the ‘holes’ is examined, rather than that of the particles.

In the paper, [6], we use steepest descent techniques to examine the edge asymptotic behaviour for cases (7) and (9). In these cases, $t \in R_1 \cup R_2$ is a root of f'_t of multiplicity 2, and the edge curve behaves like an algebraic cusp of first order. Normally, edge universality implies the *Pearcey* point process at cusps, but this does not occur in these cases. As stated at the end of section 1.2, we obtain a novel point process, which we call the *Cusp-Airy* process.

Finally, we consider some examples of the measure, μ , of hypothesis 1.1. Letting $\varphi : \mathbb{R} \rightarrow [0, 1]$ denote the density of μ , we consider:

- (a) $\varphi(x) = \frac{1}{2}$ for all $x \in [-1, 1]$.
- (b) $\varphi(x) = \frac{1}{2}$ for all $x \in [0, 1] \cup [2, 3]$.
- (c) $\varphi(x) = 1$ for all $x \in [0, \frac{1}{2}] \cup [1, \frac{3}{2}]$.
- (d) $\varphi(x) = 1$ for all $x \in [0, \frac{1}{3}] \cup [1, \frac{4}{3}] \cup [c, c + \frac{1}{3}]$, where $c := \frac{1}{12}(23 + \sqrt{217}) > \frac{4}{3}$.
- (e) $\varphi(x) = 1 - x$ for all $x \in [0, 1]$, and $\varphi(x) = 1 + x$ for all $x \in [-1, 0]$.
- (f) $\varphi(x) = \frac{15}{16}(x-1)^2(x+1)^2$ for all $x \in [-1, 1]$.

For all other values of $x \in \mathbb{R}$ in the above examples, we define $\varphi(x) := 0$. These examples are examined in detail in section 2.5. For each example, we give explicit expressions for the edge curve (see definition 1.4). Moreover, we identify those sections of $\partial\mathcal{L}$ obtained from the results of section 2.2. Finally, we identify which of the exhaustive cases, (1-9) above, exist for each example. We summarise the results of section 2.5, below.

Consider example (a). We state that the results of section 2.2 give a complete description of $\partial\mathcal{L}$ in this case. This is depicted on the top left of figure 8. Moreover, the edge is that part of $\partial\mathcal{L}$ excluding the (closed) straight line between $(-1, 1)$ and $(1, 1)$, and the point of tangency with the lower boundary. Finally, all points of the edge satisfy case (1) of (1-9). See section 2.5.1 for more details, and figure 9 for a more detailed depiction of $\partial\mathcal{L}$. Example (a) arises, for example, when we restrict n in hypothesis 1.1 to be odd, and take

$$x^{(n)} := \{n-1, n-3, \dots, 2, 0, -2, \dots, -(n-3), -(n-1)\},$$

for all such n . In words, every second particle position in the top row of the Gelfand-Tsetlin patterns is occupied.

Consider example (b). We state that the results of section 2.2 give a complete description of $\partial\mathcal{L}$ in this case. This is depicted on the top right of figure 8. Moreover, the edge is that part of $\partial\mathcal{L}$ excluding the (closed) straight line between $(0, 1)$ and $(1, 1)$, the (closed) straight line between $(2, 1)$ and $(3, 1)$, and the point of tangency with the lower boundary. Finally, the cusps in the edge satisfy case (2) of (1-9), the point of tangency with the upper boundary satisfies case (5), and all other points of the edge satisfy case (1). See section 2.5.2 for more details, and figure 10 for a more detailed depiction of $\partial\mathcal{L}$.

Consider example (c). We state that the results of section 2.2 give a complete description of $\partial\mathcal{L}$ in this case. This is depicted in the middle left of figure 8. Moreover, the edge is that part of $\partial\mathcal{L}$ excluding only the point of tangency with the lower boundary. Finally, the edge contains examples from cases (1,3,5,6,8). See section 2.5.3 for more details, and figure 11 for a more detailed depiction of $\partial\mathcal{L}$. Example (c) arises, for example, when we restrict n in hypothesis 1.1 to be even, and take

$$x^{(n)} := \left\{ \frac{3n}{2}, \frac{3n}{2} - 1, \frac{3n}{2} - 2, \dots, n+1 \right\} \cup \left\{ \frac{n}{2}, \frac{n}{2} - 1, \frac{n}{2} - 2, \dots, 1 \right\},$$

for all such n . In words, the particles on the top row of the Gelfand-Tsetlin patterns exist in 2 densely packed blocks. This is the situation for the random systems of Gelfand-Tsetlin patterns which equivalently describe random tilings of the regular hexagon (see sections 1.2 and 1.4). Finally note that the shifted asymptotic shape of the frozen boundary of the regular hexagon (see right of figure 6) is identical to $\partial\mathcal{L}$ as shown below.

Consider example (d). We state that the results of section 2.2 give a complete description of $\partial\mathcal{L}$ in this case. This is depicted in the middle right of figure 8. Moreover, the edge is that part of $\partial\mathcal{L}$ excluding only the point of tangency with the lower boundary. Finally, the edge contains examples from cases (1,3,5,6,7,8). See section 2.5.4 for more details, and figure 12 for a more detailed depiction of $\partial\mathcal{L}$. In particular, we emphasise that the cusp in the edge satisfies case (7) of (1-9). Thus, as stated above, the asymptotic

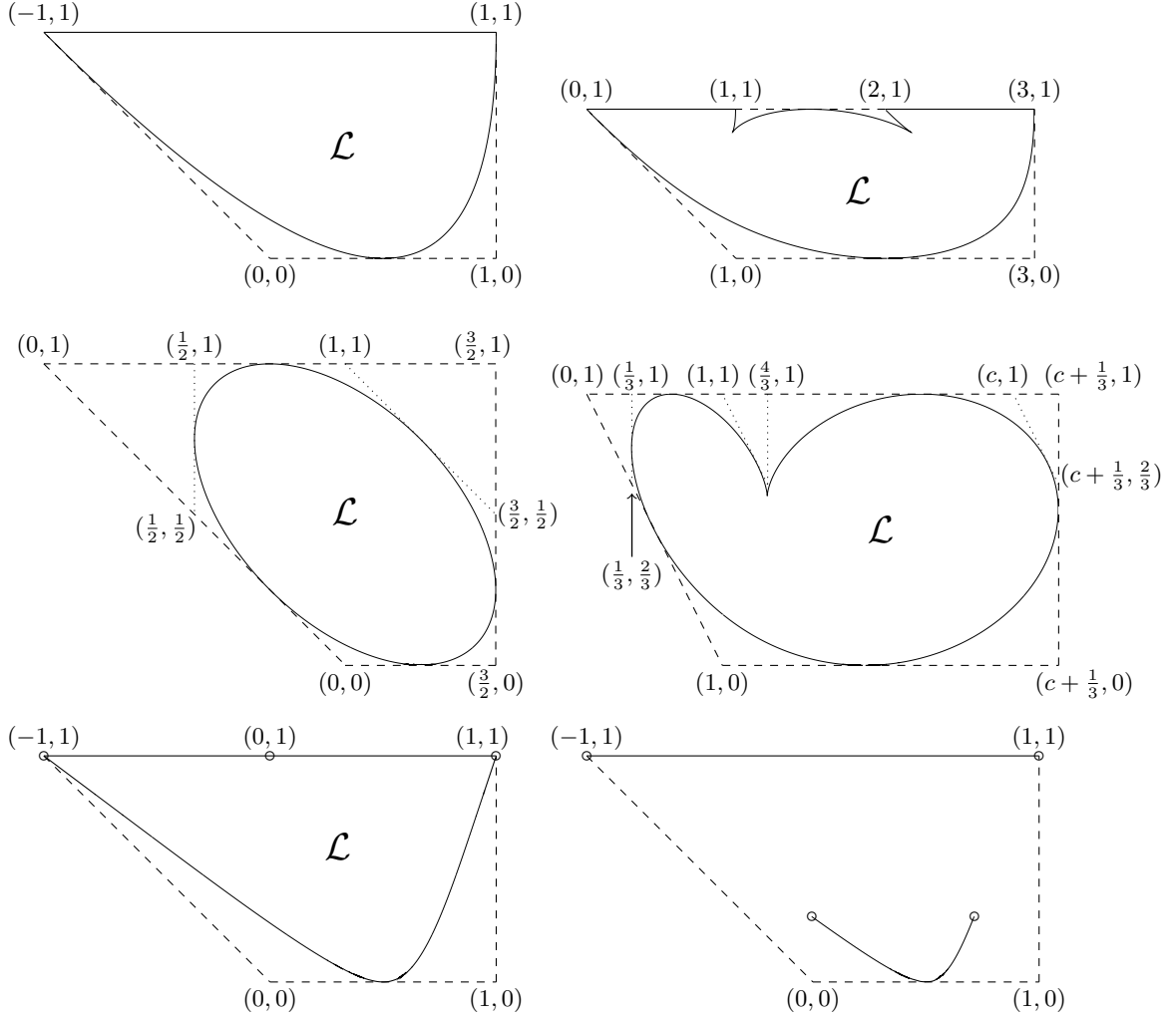


FIGURE 8. Those sections of $\partial\mathcal{L}$ obtained from the results of section 2.2 for the examples, (a-f), defined above. The dashed lines represent the shape in the middle of figure 6, and the solid lines represent $\partial\mathcal{L}$. Example (a) is on the top left, (b) is on the top right, (c) is in the middle left, (d) is in the middle right, (e) is on the bottom left, and (f) is on the bottom right. In example (d), the vertical direction has been scaled by 2 for clarity.

behaviour at this cusp is not governed by the Pearcey point process, but by the novel *Cusp-Airy* process. We prove this result in the paper [6].

Consider example (e). We state that the results of section 2.2 do not give a complete description of $\partial\mathcal{L}$ in this case. More exactly, the points $\{-1, 0, 1\} \subset \text{Supp}(\mu)$ do not satisfy any of the conditions of lemma 2.4. Consequently, we show in section 2.5.5 that the results of section 2.2 give those parts of $\partial\mathcal{L}$ shown on the bottom left of figure 8, excluding the circled points. We then show via direct calculation that we get a complete

description of $\partial\mathcal{L}$ by adding these points. The edge is that part of $\partial\mathcal{L}$ excluding the (closed) straight line between $(-1, 1)$ and $(1, 1)$, and the point of tangency with the lower boundary. Finally, all points of the edge satisfy case (1). See section 2.5.5 for more details, and figure 13 for a more detailed depiction of $\partial\mathcal{L}$.

Consider example (f). We state that the results of section 2.2 do not give a complete description of $\partial\mathcal{L}$ in this case. More exactly, the points $\{-1, 1\} \subset \text{Supp}(\mu)$ do not satisfy any of the conditions of lemma 2.4. Consequently, we show in section 2.5.6 that the results of section 2.2 give only those parts of $\partial\mathcal{L}$ shown on the bottom right of figure 8, excluding the circled points. This is clearly not a complete description of $\partial\mathcal{L}$, since \mathcal{L} is a connected set. A complete description of $\partial\mathcal{L}$, in this case, is beyond the scope of this paper, since the technicalities involved in extending lemma 2.4 to cover this situation are highly non-trivial. In the paper [4], we make heavy use of the theory of singular integrals to examine this and other, surprisingly subtle, situations. The edge, in this case, is the lower part of $\partial\mathcal{L}$ excluding the circled points, and the point of tangency with the lower boundary. Moreover, all points of the edge satisfy case (1). See section 2.5.6 for more details, and figure 14 for a more detailed depiction of $\partial\mathcal{L}$.

We end this section by noting that none of the above examples have points of the edge which satisfy either cases (4) or (9) of the exhaustive cases, (1-9), listed above. However, we note that case (4) is of a similar nature to case (2), and case (9) is of a similar nature to case (7).

2. Geometry

In this section we consider the geometric properties of the liquid region given in definition 1.3.

2.1. The liquid region, \mathcal{L} . Recall (see definition 1.3) that the liquid region, \mathcal{L} , is the set of all $(\chi, \eta) \in [a, b] \times [0, 1]$ with $b \geq \chi \geq \chi + \eta - 1 \geq a$, for which the following function has non-real roots (see equation (6)):

$$(9) \quad f'_{(\chi, \eta)}(w) = C(w) + \log(w - \chi) - \log(w - \chi - \eta + 1),$$

for all $w \in \mathbb{C} \setminus \mathbb{R}$, where \log is principal value and C is the Cauchy transform of μ ,

$$(10) \quad C(w) := \int_a^b \frac{\mu[dx]}{w - x}.$$

We denote this simply by f' where no confusion is possible. First note that non-real roots of f' occur in complex conjugate pairs. Theorem 3.1 then implies that $(\chi, \eta) \in \mathcal{L}$ if and only if f' has exactly 2 roots in $\mathbb{C} \setminus \mathbb{R}$, counting multiplicities. More exactly there are 2 roots of multiplicity 1, a unique root in $\mathbb{H} := \{w \in \mathbb{C} : \text{Im}(w) > 0\}$, and its complex conjugate.

THEOREM 2.1. *Let $W_{\mathcal{L}} : \mathcal{L} \rightarrow \mathbb{H}$ map $(\chi, \eta) \in \mathcal{L}$ to the corresponding root of f' in \mathbb{H} . This is a homeomorphism with inverse $w \mapsto (\chi_{\mathcal{L}}(w), \eta_{\mathcal{L}}(w))$ for all $w \in \mathbb{H}$, given by,*

$$\chi_{\mathcal{L}}(w) := w + \frac{(w - \bar{w})(e^{C(\bar{w})} - 1)}{e^{C(w)} - e^{C(\bar{w})}} \quad \text{and} \quad \eta_{\mathcal{L}}(w) := 1 + \frac{(w - \bar{w})(e^{C(w)} - 1)(e^{C(\bar{w})} - 1)}{e^{C(w)} - e^{C(\bar{w})}},$$

where \bar{w} is the complex conjugate of w .

PROOF. We first show:

- (1) \mathcal{L} is non-empty.
- (2) \mathcal{L} is open.
- (3) $W_{\mathcal{L}} : \mathcal{L} \rightarrow \mathbb{H}$ is continuous.
- (4) $W_{\mathcal{L}} : \mathcal{L} \rightarrow \mathbb{H}$ is injective.

The *invariance of domain theorem* then implies that $W_{\mathcal{L}}(\mathcal{L})$ is open and $W_{\mathcal{L}} : \mathcal{L} \rightarrow W_{\mathcal{L}}(\mathcal{L})$ is a homeomorphism. We complete the result by showing:

- (5) $W_{\mathcal{L}} : \mathcal{L} \rightarrow W_{\mathcal{L}}(\mathcal{L})$ has inverse $w \mapsto (\chi_{\mathcal{L}}(w), \eta_{\mathcal{L}}(w))$ for all $w \in W_{\mathcal{L}}(\mathcal{L})$.
- (6) $W_{\mathcal{L}}(\mathcal{L}) = \mathbb{H}$.

Consider (1). Fixing $w \in \mathbb{H}$ and defining $(\chi, \eta) := (\chi_{\mathcal{L}}(w), \eta_{\mathcal{L}}(w))$, where $\chi_{\mathcal{L}}$ and $\eta_{\mathcal{L}}$ are defined as in the statement of the lemma, we show that:

- (1a) $f'(w) = 0$.
- (1b) $(\chi, \eta) \in [a, b] \times [0, 1]$ with $b \geq \chi \geq \chi + \eta - 1 \geq a$ whenever $|w|$ is chosen to be sufficiently large.

Thus $(\chi, \eta) \in \mathcal{L}$ by definition whenever $|w|$ is chosen to be sufficiently large, as required.

Consider (1a). First note, since $(\chi, \eta) = (\chi_{\mathcal{L}}(w), \eta_{\mathcal{L}}(w))$, the definitions of $\chi_{\mathcal{L}}$ and $\eta_{\mathcal{L}}$ give $w - \chi - \eta + 1 = (w - \chi)e^{C(w)}$. Equation (9) then gives $f'(w) = C(w) + \log(w - \chi) - \log((w - \chi)e^{C(w)})$, where \log is principal value. (1a) thus follows if we can show that $\log((w - \chi)e^{C(w)}) = \log(w - \chi) + \log(e^{C(w)})$. We prove this by showing that $\text{Arg}(w - \chi) \in (0, \pi)$ and $\text{Arg}(e^{C(w)}) \in (-\pi, 0)$.

First note that it is trivial to see that $\text{Arg}(w - \chi) \in (0, \pi)$, since $w \in \mathbb{H}$ and $\chi \in \mathbb{R}$. Next note that equation (10) gives,

$$\text{Im}(C(w)) = - \int_a^b \frac{\text{Im}(w)\mu[dx]}{(\text{Re}(w) - x)^2 + \text{Im}(w)^2}.$$

Thus, since $\text{Im}(w) > 0$, and since $\mu \leq \lambda$ (see remark 1.1),

$$0 > \text{Im}(C(w)) > - \int_{-\infty}^{\infty} \frac{\text{Im}(w)dx}{(\text{Re}(w) - x)^2 + \text{Im}(w)^2} = -\pi.$$

Therefore $\text{Arg}(e^{C(w)}) = \text{Im}(C(w)) \in (-\pi, 0)$. This proves (1a).

Consider (1b). Recalling that $\chi = \chi_{\mathcal{L}}(w)$ and $\eta = \eta_{\mathcal{L}}(w)$, write

$$\chi = \frac{w(e^{\frac{1}{2}(C(w)-C(\bar{w}))} - 1) - \bar{w}(e^{-\frac{1}{2}(C(w)-C(\bar{w}))} - 1) - (w - \bar{w})(e^{-\frac{1}{2}(C(w)+C(\bar{w}))} - 1)}{2 \sinh(\frac{1}{2}(C(w) - C(\bar{w})))},$$

$$\eta = 1 + \frac{2(w - \bar{w}) \sinh(\frac{1}{2}C(w)) \sinh(\frac{1}{2}C(\bar{w}))}{\sinh(\frac{1}{2}(C(w) - C(\bar{w})))}.$$

Also Taylor expansions of the Cauchy transform in equation (10) give

$$C(w) = \frac{1}{w} + \frac{\mu_1}{w^2} + \frac{\mu_2}{w^3} + O(|w|^{-4}),$$

$$C(w) - C(\bar{w}) = \left(\frac{1}{w} - \frac{1}{\bar{w}} \right) \left(1 + \mu_1 \left(\frac{1}{w} + \frac{1}{\bar{w}} \right) + \mu_2 \left(\frac{1}{w^2} + \frac{1}{|w|^2} + \frac{1}{\bar{w}^2} \right) + O(|w|^{-3}) \right),$$

where $\mu_1 := \int_a^b x \mu[dx]$ and $\mu_2 := \int_a^b x^2 \mu[dx]$. Combine the above to get

$$\chi = \mu_1 + \frac{1}{2} + O(|w|^{-1}) \quad \text{and} \quad \eta = \left(\mu_2 - \mu_1^2 - \frac{1}{12} \right) \frac{1}{|w|^2} + O(|w|^{-3}).$$

Finally recall that hypothesis 1.1 and remark 1.1 imply that μ is a probability measure on $[a, b]$, that $b - a > 1$, and that $\mu \leq \lambda$. Also $a \in \text{Supp}(\mu)$, and so $\mu[a, a + \epsilon] > 0$ for all $\epsilon > 0$. Therefore,

$$\mu_1 = \int_a^b x \mu[dx] < \int_{b-1}^b x dx = b - \frac{1}{2}.$$

Similarly $b \in \text{Supp}(\mu)$, and so

$$\mu_1 = \int_a^b x \mu[dx] > \int_a^{a+1} x dx = a + \frac{1}{2}.$$

Similarly,

$$\mu_2 - \mu_1^2 = \frac{1}{2} \iint_a^b (x - y)^2 \mu[dx] \mu[dy] > \frac{1}{2} \iint_0^1 (x - y)^2 dx dy = \frac{1}{12}.$$

Combine the above to get $(\chi, \eta) \in (a+1, b) \times (0, 1)$ whenever $|w|$ is chosen to be sufficiently large. This proves (1b).

Consider (2). Fix $(\chi_1, \eta_1) \in \mathcal{L}$ and $(\chi_2, \eta_2) \in [a, b] \times [0, 1]$ and define

- $f'_1(w) := C(w) + \log(w - \chi_1) - \log(w - \chi_1 - \eta_1 + 1)$,
- $f'_2(w) := C(w) + \log(w - \chi_2) - \log(w - \chi_2 - \eta_2 + 1)$,

for all $w \in \mathbb{C} \setminus \mathbb{R}$, where \log is principal value. Let $w_1 := W_{\mathcal{L}}(\chi_1, \eta_1)$, where $W_{\mathcal{L}}$ is defined in the statement of the lemma. Then w_1 is the unique root of f'_1 in \mathbb{H} . We now show that $(\chi_2, \eta_2) \in \mathcal{L}$ whenever $|\chi_1 - \chi_2|$ and $|\eta_1 - \eta_2|$ are sufficiently small, i.e., that f'_2 has a root in \mathbb{H} for all such (χ_2, η_2) . Fix $\epsilon > 0$ for which $B(w_1, \epsilon) \subset \mathbb{H}$. Then, since w_1 is the unique root of f'_1 in \mathbb{H} , the extreme value theorem gives,

$$\inf_{w \in \partial B(w_1, \epsilon)} |f'_1(w)| > 0.$$

Also $|f'_1(w) - f'_2(w)| \leq |\log(w - \chi_1) - \log(w - \chi_2)| + |\log(w - \chi_1 - \eta_1 + 1) - \log(w - \chi_2 - \eta_2 + 1)|$. Thus, whenever $|\chi_1 - \chi_2|$ and $|\eta_1 - \eta_2|$ are sufficiently small, $|f'_1(w)| > |f'_1(w) - f'_2(w)|$ for all $w \in \partial B(w_1, \epsilon)$. Rouché's Theorem thus implies that f'_2 has a root in $B(w_1, \epsilon) \subset \mathbb{H}$.

Consider (3). Consider the same setup used in step (2). Recall that $w_1 = W_{\mathcal{L}}(\chi_1, \eta_1)$ and that f'_2 has a root in $B(w_1, \epsilon) \subset \mathbb{H}$ whenever $|\chi_1 - \chi_2|$ and $|\eta_1 - \eta_2|$ are sufficiently small. The above definition of $W_{\mathcal{L}}$ then implies that $W_{\mathcal{L}}(\chi_2, \eta_2)$ is the observed root of f'_2 , and so $|W_{\mathcal{L}}(\chi_1, \eta_1) - W_{\mathcal{L}}(\chi_2, \eta_2)| < \epsilon$ whenever $|\chi_1 - \chi_2|$ and $|\eta_1 - \eta_2|$ are sufficiently

small. Finally note that we can repeat the same analysis with ϵ replaced by any $\delta \in (0, \epsilon)$. Therefore $W_{\mathcal{L}}$ is continuous, as required.

Consider (4). Fix $(\chi_1, \eta_1), (\chi_2, \eta_2) \in \mathcal{L}$ with $W_{\mathcal{L}}(\chi_1, \eta_1) = W_{\mathcal{L}}(\chi_2, \eta_2) = w \in \mathbb{H}$. Equation (9) and the above definition of $W_{\mathcal{L}}$ then gives

$$C(w) = -\log(w - \chi_1) + \log(w - \chi_1 - \eta_1 + 1) = -\log(w - \chi_2) + \log(w - \chi_2 - \eta_2 + 1).$$

Exponentiating and simplifying gives $(\eta_2 - \eta_1)w = (1 - \eta_1)\chi_2 - (1 - \eta_2)\chi_1$. Then $w \in \mathbb{R}$ whenever $\eta_1 \neq \eta_2$, which contradicts $w \in \mathbb{H}$. Thus $\eta_1 = \eta_2$. Also, recalling that \mathcal{L} is open, $\eta_1 = \eta_2 \in (0, 1)$. This finally gives $\chi_1 = \chi_2$, as required.

Consider (5). Fix $(\chi, \eta) \in \mathcal{L}$ and let $w := W_{\mathcal{L}}(\chi, \eta) \in W_{\mathcal{L}}(\mathcal{L})$. Equation (9) and the above definition of $W_{\mathcal{L}}$ then give $C(w) + \log(w - \chi) - \log(w - \chi - \eta + 1) = 0$. Exponentiating and simplifying gives $1 - \eta = (w - \chi)(e^{C(w)} - 1)$. Complex conjugation gives,

$$1 - \eta = (w - \chi)(e^{C(w)} - 1) = (\bar{w} - \chi)(e^{C(\bar{w})} - 1).$$

Solving gives $(\chi, \eta) = (\chi_{\mathcal{L}}(w), \eta_{\mathcal{L}}(w))$, as required.

Consider (6). Recall that $W_{\mathcal{L}}(\mathcal{L})$ is open and that $W_{\mathcal{L}} : \mathcal{L} \rightarrow W_{\mathcal{L}}(\mathcal{L})$ is a homeomorphism with inverse $w \mapsto (\chi_{\mathcal{L}}(w), \eta_{\mathcal{L}}(w))$. Assume that $W_{\mathcal{L}}(\mathcal{L})$ is a proper subset of \mathbb{H} , i.e., that there exists a point $w \in \partial W_{\mathcal{L}}(\mathcal{L})$ with $w \in \mathbb{H} \setminus W_{\mathcal{L}}(\mathcal{L})$. Choose a sequence $\{w_n\}_{n \geq 1} \subset W_{\mathcal{L}}(\mathcal{L})$ with $w_n \rightarrow w$ as $n \rightarrow \infty$, and let $(\chi_n, \eta_n) := (\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n))$ for all $n \geq 1$. Note that we can always choose so that $\{(\chi_n, \eta_n)\}_{n \geq 1}$ is convergent as $n \rightarrow \infty$, $(\chi_n, \eta_n) \rightarrow (\chi, \eta)$ say. Also note equation (9) and the above definition of $W_{\mathcal{L}}$ gives $C(w_n) + \log(w_n - \chi_n) - \log(w_n - \chi_n - \eta_n + 1) = 0$ for all $n \geq 1$. Letting $n \rightarrow \infty$ we get $C(w) + \log(w - \chi) - \log(w - \chi - \eta + 1) = 0$, and so $(\chi, \eta) \in \mathcal{L}$ and $w = W_{\mathcal{L}}(\chi, \eta)$. This contradicts the assumption that $w \in \mathbb{H} \setminus W_{\mathcal{L}}(\mathcal{L})$, and so $W_{\mathcal{L}}(\mathcal{L}) = \mathbb{H}$, as required. \square

It can furthermore be shown that $W_{\mathcal{L}} : \mathcal{L} \rightarrow \mathbb{H}$ is a diffeomorphism (see section 4.2).

2.2. The boundary of the liquid region, $\partial\mathcal{L}$. Note the following consequence of theorem 2.1:

COROLLARY 2.2. *\mathcal{L} is a non-empty, open, simply connected set in the shape in the middle of figure 6. Moreover, $\partial\mathcal{L}$ is the set of all (χ, η) for which there exists a sequence, $\{w_n\}_{n \geq 1} \subset \mathbb{H}$, with $(\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n)) \rightarrow (\chi, \eta)$ as $n \rightarrow \infty$, and either $|w_n| \rightarrow \infty$ or $w_n \rightarrow t \in \mathbb{R}$ as $n \rightarrow \infty$.*

In this section we use the above result to examine $\partial\mathcal{L}$. First note:

LEMMA 2.1. *$(\frac{1}{2} + \int_a^b x\mu[dx], 0) \in \partial\mathcal{L}$. Moreover $(\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n)) \rightarrow (\frac{1}{2} + \int_a^b x\mu[dx], 0)$ as $n \rightarrow \infty$ for all $\{w_n\}_{n \geq 1} \subset \mathbb{H}$ with $|w_n| \rightarrow \infty$.*

PROOF. Fix $\{w_n\}_{n \geq 1} \subset \mathbb{H}$ with $|w_n| \rightarrow \infty$ as $n \rightarrow \infty$. The proof of step (1b) of theorem 2.1 then gives $(\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n)) \rightarrow (\frac{1}{2} + \int_a^b x\mu[dx], 0)$, and corollary 2.2 gives $(\frac{1}{2} + \int_a^b x\mu[dx], 0) \in \partial\mathcal{L}$. \square

Next recall that $\mu \leq \lambda$ (see remark 1.1), and note that $\mathbb{R} \setminus \text{Supp}(\mu)$ and $\mathbb{R} \setminus \text{Supp}(\lambda - \mu)$ are disjoint open sets. Also recall (see equation (8)) that

$$R := (\overline{(\mathbb{R} \setminus \text{Supp}(\mu)) \cup (\mathbb{R} \setminus \text{Supp}(\lambda - \mu))})^\circ.$$

Hypothesis 1.1 and remark 1.1 then give,

$$(11) \quad R = (\mathbb{R} \setminus \text{Supp}(\mu)) \cup (\mathbb{R} \setminus \text{Supp}(\lambda - \mu)) \cup R_1 \cup R_2,$$

where,

- R_1 is the set of all $t \in \partial(\mathbb{R} \setminus \text{Supp}(\mu)) \cap \partial(\mathbb{R} \setminus \text{Supp}(\lambda - \mu))$ for which there exists an interval, $I := (t_2, t_1)$, with $t \in I$, $(t, t_1) \subset \mathbb{R} \setminus \text{Supp}(\mu)$ and $(t_2, t) \subset \mathbb{R} \setminus \text{Supp}(\lambda - \mu)$.
- R_2 is the set of all $t \in \partial(\mathbb{R} \setminus \text{Supp}(\mu)) \cap \partial(\mathbb{R} \setminus \text{Supp}(\lambda - \mu))$ for which there exists an interval, $I := (t_2, t_1)$, with $t \in I$, $(t, t_1) \subset \mathbb{R} \setminus \text{Supp}(\lambda - \mu)$ and $(t_2, t) \subset \mathbb{R} \setminus \text{Supp}(\mu)$.

Indeed, $R_1 \cap R_2 = \emptyset$ and $R_1 \cup R_2 = \partial(\mathbb{R} \setminus \text{Supp}(\mu)) \cap \partial(\mathbb{R} \setminus \text{Supp}(\lambda - \mu))$. Finally, note the following technical result:

LEMMA 2.2. *Let $C : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}$ denote the Cauchy transform of μ (see equation (10)). Then,*

- (a) *$C : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}$ has a unique analytic extension to $\mathbb{C} \setminus \text{Supp}(\mu)$. Moreover, also denoting the extension by C ,*

$$C(w) = \int_a^b \frac{\mu[dx]}{w - x},$$

- for all $w \in \mathbb{C} \setminus \text{Supp}(\mu)$. Finally, $e^{C(t)} > 0$ and $C'(t) < 0$ for all $t \in \mathbb{R} \setminus \text{Supp}(\mu)$.*
- (b) *$e^{C(\cdot)} : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}$ and $C' : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}$ have unique analytic extensions to $\mathbb{C} \setminus \text{Supp}(\lambda - \mu)$. Moreover, also denoting the extensions by $e^{C(\cdot)}$ and C' , and fixing any interval $I = (t_2, t_1) \subset \mathbb{R} \setminus \text{Supp}(\lambda - \mu)$,*

$$e^{C(w)} = e^{C_I(w)} \left(\frac{w - t_2}{w - t_1} \right) \quad \text{and} \quad C'(w) = C'_I(w) - \frac{1}{w - t_1} + \frac{1}{w - t_2},$$

- for all $w \in (\mathbb{C} \setminus \mathbb{R}) \cup I$, where $C_I(w) := \int_{[a,b] \setminus I} \frac{\mu[dx]}{w - x}$. Finally, $e^{C(t)} < 0$, $C'(t) > 0$ and $\frac{d}{dw} e^{C(w)}|_{w=t} = e^{C(t)} C'(t)$ for all $t \in \mathbb{R} \setminus \text{Supp}(\lambda - \mu)$.*
- (c) *Fix $t \in R_1$, and $I = (t_2, t_1)$ with $t \in I$, $(t, t_1) \subset \mathbb{R} \setminus \text{Supp}(\mu)$ and $(t_2, t) \subset \mathbb{R} \setminus \text{Supp}(\lambda - \mu)$ (see equation (11)). Then $e^{-C(\cdot)} : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}$ has a unique analytic extension to $(\mathbb{C} \setminus \mathbb{R}) \cup I$. Moreover, also denoting the extension by $e^{-C(\cdot)}$,*

$$e^{-C(w)} = e^{-C_I(w)} \left(\frac{w - t}{w - t_2} \right),$$

for all $w \in (\mathbb{C} \setminus \mathbb{R}) \cup I$, where $C_I(w) := \int_{[a,b] \setminus I} \frac{\mu[dx]}{w - x}$. Finally, $e^{-C(t)} = 0$ and $\frac{d}{dw} e^{-C(w)}|_{w=t} = e^{-C_I(t)}(t - t_2)^{-1}$.

- (d) Fix $t \in R_2$, and $I = (t_2, t_1)$ with $t \in I$, $(t, t_1) \subset \mathbb{R} \setminus \text{Supp}(\lambda - \mu)$ and $(t_2, t) \subset \mathbb{R} \setminus \text{Supp}(\mu)$ (see equation (11)). Then $e^{C(\cdot)} : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}$ has a unique analytic extension to $(\mathbb{C} \setminus \mathbb{R}) \cup I$. Moreover, also denoting the extension by $e^{C(\cdot)}$,

$$e^{C(w)} = e^{C_I(w)} \left(\frac{w - t}{w - t_1} \right),$$

for all $w \in (\mathbb{C} \setminus \mathbb{R}) \cup I$, where $C_I(w) := \int_{[a,b] \setminus I} \frac{\mu[dx]}{w-x}$. Finally, $e^{C(t)} = 0$ and $\frac{d}{dw} e^{C(w)}|_{w=t} = e^{C_I(t)}(t - t_1)^{-1}$.

PROOF. Consider (a). The required analytic extension easily follows from equation (10). Also,

$$C(w) = \int_a^b \frac{\mu[dx]}{w-x} \quad \text{and} \quad C'(w) = - \int_a^b \frac{\mu[dx]}{(w-x)^2},$$

for all $w \in \mathbb{C} \setminus \text{Supp}(\mu)$. Thus $C(t) \in \mathbb{R}$, $e^{C(t)} > 0$ and $C'(t) < 0$ for all $t \in \mathbb{R} \setminus \text{Supp}(\mu)$.

Consider (b). Fixing $I = (t_2, t_1) \subset \mathbb{R} \setminus \text{Supp}(\lambda - \mu)$, equation (10) gives $C(w) = C_I(w) - \log(w - t_1) + \log(w - t_2)$ for all $w \in \mathbb{C} \setminus \mathbb{R}$, where \log is principal value and $C_I(w) = \int_{[a,b] \setminus I} \frac{\mu[dx]}{w-x}$. The required analytic extensions easily follow. Also, it easily follows from the expressions of the extensions that $e^{C(t)} < 0$ and $\frac{d}{dw} e^{C(w)}|_{w=t} = e^{C(t)} C'(t)$ for all $t \in I$. It remains to show that $C'(t) < 0$ for all $t \in I$. Note, for all such t ,

$$C'(t) = - \left(\int_a^{t_2} + \int_{t_1}^b \right) \frac{\mu[dx]}{(t-x)^2} - \frac{1}{t-t_1} + \frac{1}{t-t_2}.$$

Then, recalling that $\mu \leq \lambda$ (see remark 1.1),

$$C'(t) \geq - \left(\int_{t_2-p_2}^{t_2} + \int_{t_1}^{t_1+p_1} \right) \frac{dx}{(t-x)^2} - \frac{1}{t-t_1} + \frac{1}{t-t_2},$$

where $p_1 := \mu[t_1, b]$ and $p_2 := \mu[a, t_2]$. Integrating finally gives,

$$C'(t) \geq \frac{1}{t-t_2+p_2} - \frac{1}{t-t_1-p_1} = \frac{-1}{(t-t_2+p_2)(t-t_1-p_1)},$$

where the final part follows since $p_1 + p_2 + (t_1 - t_2) = \mu[t_1, b] + \mu[a, t_2] + \mu[t_2, t_1] = 1$. Therefore $C'(t) > 0$ for all $t \in I = (t_2, t_1)$.

Consider (c). Fixing $t \in R_1$ and $I = (t_2, t_1)$ as in the statement, equation (10) gives $C(w) = C_I(w) - \log(w - t) + \log(w - t_2)$ for all $w \in \mathbb{C} \setminus \mathbb{R}$, where \log is principal value. The required analytic extension easily follows. Also, it easily follows from the expression of the extension that $e^{-C(t)} = 0$ and $\frac{d}{dw} e^{-C(w)}|_{w=t} = e^{-C_I(t)}(t - t_2)^{-1}$. Similarly for (d). \square

The various analytic extensions of the previous result then give:

LEMMA 2.3. $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) \in \partial \mathcal{L}$ for all $t \in (\mathbb{R} \setminus \text{Supp}(\mu)) \cup (\mathbb{R} \setminus \text{Supp}(\lambda - \mu))$, where

$$\chi_{\mathcal{E}}(t) := t + \frac{e^{C(t)} - 1}{e^{C(t)} C'(t)} \quad \text{and} \quad \eta_{\mathcal{E}}(t) := 1 + \frac{(e^{C(t)} - 1)^2}{e^{C(t)} C'(t)}.$$

Also $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) \in \partial \mathcal{L}$ for all $t \in R_1 \cup R_2$, where

- $\chi_{\mathcal{E}}(t) := t$ and $\eta_{\mathcal{E}}(t) := 1 - e^{C_I(t)}(t - t_2)$ for all $t \in R_1$,
- $\chi_{\mathcal{E}}(t) := t - e^{-C_I(t)}(t - t_1)$ and $\eta_{\mathcal{E}}(t) := 1 + e^{-C_I(t)}(t - t_1)$ for all $t \in R_2$.

Moreover, $(\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n)) \rightarrow (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ as $n \rightarrow \infty$ for all $t \in R$ and $\{w_n\}_{n \geq 1} \subset \mathbb{H}$ with $w_n \rightarrow t$. (Above, whenever $t \in R_1 \cup R_2$, $I := (t_2, t_1)$ is chosen as in lemma 2.2, $C_I(w) := \int_{[a,b] \setminus I} \frac{\mu[dx]}{w-x}$, and the result is independent of the choice of I .)

PROOF. Fix $t \in R$ and $\{w_n\}_{n \geq 1} \subset \mathbb{H}$ with $w_n \rightarrow t$ as $n \rightarrow \infty$. We shall show that $(\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n)) \rightarrow (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ as $n \rightarrow \infty$. Corollary 2.2 then gives $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) \in \partial \mathcal{L}$, as required.

When $t \in \mathbb{R} \setminus \text{Supp}(\mu)$, write (see theorem 2.1),

$$\begin{aligned}\chi_{\mathcal{L}}(w_n) &= w_n + (e^{C(w_n)} - 1) \frac{w_n - \overline{w_n}}{e^{C(w_n)} - e^{C(\overline{w_n})}}, \\ \eta_{\mathcal{L}}(w_n) &= 1 + (e^{C(w_n)} - 1)(e^{C(\overline{w_n})} - 1) \frac{w_n - \overline{w_n}}{e^{C(w_n)} - e^{C(\overline{w_n})}},\end{aligned}$$

for all n . Note, as $n \rightarrow \infty$, the analytic extension of part (a) of lemma 2.2 gives,

$$e^{C(w_n)} \rightarrow e^{C(t)}, \quad e^{C(\overline{w_n})} \rightarrow e^{C(t)} \quad \text{and} \quad \frac{e^{C(w_n)} - e^{C(\overline{w_n})}}{w_n - \overline{w_n}} \rightarrow e^{C(t)} C'(t).$$

Therefore $(\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n)) \rightarrow (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ when $t \in \mathbb{R} \setminus \text{Supp}(\mu)$. Similarly when $t \in \mathbb{R} \setminus \text{Supp}(\lambda - \mu)$, except now we use the analytic extensions of part (b) of lemma 2.2.

Also write $\chi_{\mathcal{L}}(w_n)$ and $\eta_{\mathcal{L}}(w_n)$ as above when $t \in R_2$. When $t \in R_1$, write,

$$\begin{aligned}\chi_{\mathcal{L}}(w_n) &= w_n - e^{-C(w_n)}(1 - e^{-C(\overline{w_n})}) \frac{w_n - \overline{w_n}}{e^{-C(w_n)} - e^{-C(\overline{w_n})}}, \\ \eta_{\mathcal{L}}(w_n) &= 1 - (1 - e^{-C(w_n)})(1 - e^{-C(\overline{w_n})}) \frac{w_n - \overline{w_n}}{e^{-C(w_n)} - e^{-C(\overline{w_n})}},\end{aligned}$$

for all n . The analytic extensions of parts (c) and (d) of lemma 2.2 then easily give $(\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n)) \rightarrow (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ as $n \rightarrow \infty$ when $t \in R_1 \cup R_2$, as required. \square

REMARK 2.1. For emphasis we note that $R \subset \mathbb{R}$ is an open set, and that lemma 2.3 implies that $(\chi_{\mathcal{E}}(\cdot), \eta_{\mathcal{E}}(\cdot)) : R \rightarrow \partial \mathcal{L}$ is the unique continuous extensions to R of $(\chi_{\mathcal{L}}(\cdot), \eta_{\mathcal{L}}(\cdot)) : \mathbb{H} \rightarrow \mathcal{L}$. Thus $(\chi_{\mathcal{E}}(\cdot), \eta_{\mathcal{E}}(\cdot))$ is a smooth curve parameterised over R . In definition 1.4 we defined the edge, $\mathcal{E} \subset \partial \mathcal{L}$, as the image of this curve. This is why we have chosen the subscript \mathcal{E} . Definition 1.5 equivalently defines the edge in terms of the behaviour of the roots of the function f' of equation (7). This equivalence is shown in corollary 2.4 of theorem 2.3.

It remains to consider the asymptotic behaviour of $(\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n))$ as $n \rightarrow \infty$, when $t \in \mathbb{R} \setminus R$ and $\{w_n\}_{n \geq 1} \subset \mathbb{H}$ with $w_n \rightarrow t$. This question is surprisingly subtle however, and will be examined in greater detail in [4]. Lemma 2.4, below, is a sub-result of that paper. However, lemmas 2.1, 2.3 and 2.4 give a complete description of $\partial \mathcal{L}$ when the measure, μ , of hypothesis 1.1 is restricted to an interesting sub-class of all possible measures (see lemma 2.5). In section 2.5 we depict \mathcal{L} and $\partial \mathcal{L}$ for a number of examples in this sub-class.

LEMMA 2.4. $(t, 1) \in \partial\mathcal{L}$ for all $t \in \mathbb{R} \setminus R = \text{Supp}(\mu) \cap \text{Supp}(\lambda - \mu) \cap (\mathbb{R} \setminus (R_1 \cup R_2))$ whenever there exists an $\epsilon > 0$ for which one of the following cases is satisfied:

- (1) $\sup_{x \in (t-\epsilon, t+\epsilon)} \varphi(x) < 1$ and $\inf_{x \in (t-\epsilon, t+\epsilon)} \varphi(x) > 0$.
- (2) $\sup_{x \in (t-\epsilon, t)} \varphi(x) < 1$, $\inf_{x \in (t-\epsilon, t)} \varphi(x) > 0$ and $\varphi(x) = 0$ for all $x \in (t, t+\epsilon)$.
- (3) $\sup_{x \in (t-\epsilon, t)} \varphi(x) < 1$, $\inf_{x \in (t-\epsilon, t)} \varphi(x) > 0$ and $\varphi(x) = 1$ for all $x \in (t, t+\epsilon)$.
- (4) $\sup_{x \in (t, t+\epsilon)} \varphi(x) < 1$, $\inf_{x \in (t, t+\epsilon)} \varphi(x) > 0$ and $\varphi(x) = 0$ for all $x \in (t-\epsilon, t)$.
- (5) $\sup_{x \in (t, t+\epsilon)} \varphi(x) < 1$, $\inf_{x \in (t, t+\epsilon)} \varphi(x) > 0$ and $\varphi(x) = 1$ for all $x \in (t-\epsilon, t)$.

Above $\varphi : [a, b] \rightarrow [0, 1]$ denotes the density of μ (remark 1.1 shows that φ is well-defined). Moreover $(\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n)) \rightarrow (t, 1)$ as $n \rightarrow \infty$ for all $\{w_n\}_{n \geq 1} \subset \mathbb{H}$ with $w_n \rightarrow t$.

PROOF. Fix $t \in \mathbb{R} \setminus R$ which satisfies one of the cases (1-5) in the statement of the lemma. Also fix $\epsilon > 0$ as in the statement, and $\{w_n\}_{n \geq 1} \subset \mathbb{H}$ with $w_n \rightarrow t$ as $n \rightarrow \infty$. Denote $u_n := \text{Re}(w_n)$, $v_n := \text{Im}(w_n)$, $R_n := \text{Re}(C(w_n))$, and $I_n := -\text{Im}(C(w_n))$, where C is the Cauchy transform of μ (see equation (10)). Therefore,

$$(12) \quad R_n = \int_a^b \frac{(u_n - x)\varphi(x)dx}{(u_n - x)^2 + v_n^2} \quad \text{and} \quad I_n = \int_a^b \frac{v_n\varphi(x)dx}{(u_n - x)^2 + v_n^2},$$

for all n . Note that $u_n \rightarrow t$ and $v_n \searrow 0$ as $n \rightarrow \infty$, and that $\pi > I_n > 0$ for all n . Also write (see theorem 2.1),

$$(13) \quad \chi_{\mathcal{L}}(w_n) = u_n - \frac{v_n \cos(I_n) - v_n e^{-R_n}}{\sin(I_n)},$$

$$(14) \quad \eta_{\mathcal{L}}(w_n) = 1 - \frac{v_n e^{R_n} - 2v_n \cos(I_n) + v_n e^{-R_n}}{\sin(I_n)},$$

for all n . We shall show that $(\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n)) \rightarrow (t, 1)$ as $n \rightarrow \infty$. Corollary 2.2 then gives $(t, 1) \in \partial\mathcal{L}$, as required. We consider cases (1) and (2), and note (3-5) are similar.

Consider (1). Letting $\varphi^+ := \sup_{x \in (t-\epsilon, t+\epsilon)} \varphi(x) < 1$ and $\varphi^- := \inf_{x \in (t-\epsilon, t+\epsilon)} \varphi(x) > 0$, we show that there exists a constant $c > 0$ such that, for all n sufficiently large,

$$(1a) \quad \pi - \frac{\pi}{2}(1 - \varphi^+) > I_n > \frac{\pi}{2}\varphi^-.$$

$$(1b) \quad v_n e^{|R_n|} < c v_n^{1-\varphi^++\varphi^-}.$$

Equations (13) and (14) then easily give $(\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n)) \rightarrow (t, 1)$ as $n \rightarrow \infty$, as required.

Consider (1a). Recall that $\varphi(x) \in [0, 1]$ for all $x \in [a, b]$, $\varphi^+ = \sup_{x \in (t-\epsilon, t+\epsilon)} \varphi(x) < 1$ and $\varphi^- = \inf_{x \in (t-\epsilon, t+\epsilon)} \varphi(x) > 0$. Defining $\varphi(x) := 0$ for all $x \in \mathbb{R} \setminus [a, b]$, equation (12) then gives,

$$\begin{aligned} I_n &\leq \int_{a-\epsilon}^{t-\epsilon} \frac{v_n(1)dx}{(u_n - x)^2 + v_n^2} + \int_{t-\epsilon}^{t+\epsilon} \frac{v_n(\varphi^+)dx}{(u_n - x)^2 + v_n^2} + \int_{t+\epsilon}^{b+\epsilon} \frac{v_n(1)dx}{(u_n - x)^2 + v_n^2} \\ &= -\arctan\left(\frac{u_n - x}{v_n}\right)\Big|_{a-\epsilon}^{t-\epsilon} - \varphi^+ \arctan\left(\frac{u_n - x}{v_n}\right)\Big|_{t-\epsilon}^{t+\epsilon} - \arctan\left(\frac{u_n - x}{v_n}\right)\Big|_{t+\epsilon}^{b+\epsilon}, \end{aligned}$$

for all n . Thus, since $u_n \rightarrow t \in [a, b]$ and $v_n \searrow 0$ as $n \rightarrow \infty$, $I_n \leq \pi\varphi^+ + O(v_n)$. Similarly,

$$I_n \geq \int_{a-\epsilon}^{t-\epsilon} (0)dx + \int_{t-\epsilon}^{t+\epsilon} \frac{v_n(\varphi^-)dx}{(u_n - x)^2 + v_n^2} + \int_{t+\epsilon}^{b+\epsilon} (0)dx,$$

for all n . Proceed as before to get $I_n \geq \pi\varphi^- + O(v_n)$. (1a) follows since $\varphi^+ < 1$ and $\varphi^- > 0$.

Consider (1b). Recall that $\varphi : \mathbb{R} \rightarrow [0, 1]$, $\varphi(x) = 0$ for all $x \in \mathbb{R} \setminus [a, b]$, $\varphi^+ = \sup_{x \in (t-\epsilon, t+\epsilon)} \varphi(x) < 1$ and $\varphi^- = \inf_{x \in (t-\epsilon, t+\epsilon)} \varphi(x) > 0$. Then, choosing n sufficiently large that $u_n \in (t - \epsilon, t + \epsilon)$, equation (12) gives,

$$\begin{aligned} R_n &\leq \int_{a-\epsilon}^{t-\epsilon} \frac{(u_n - x)(1)dx}{(u_n - x)^2 + v_n^2} + \int_{t-\epsilon}^{u_n} \frac{(u_n - x)(\varphi^+)dx}{(u_n - x)^2 + v_n^2} + \int_{u_n}^{t+\epsilon} \frac{(u_n - x)(\varphi^-)dx}{(u_n - x)^2 + v_n^2} + \int_{t+\epsilon}^{b+\epsilon} (0)dx \\ &= -\frac{1}{2} \log((u_n - x)^2 + v_n^2) \Big|_{a-\epsilon}^{t-\epsilon} - \frac{\varphi^+}{2} \log((u_n - x)^2 + v_n^2) \Big|_{t-\epsilon}^{u_n} - \frac{\varphi^-}{2} \log((u_n - x)^2 + v_n^2) \Big|_{u_n}^{t+\epsilon}. \end{aligned}$$

Thus, since $u_n \rightarrow t$ and $v_n \searrow 0$ as $n \rightarrow \infty$, $R_n \leq -(\varphi^+ - \varphi^-) \log(v_n) + O(1)$. Similarly,

$$R_n \geq \int_{a-\epsilon}^{t-\epsilon} (0)dx + \int_{t-\epsilon}^{u_n} \frac{(u_n - x)(\varphi^-)dx}{(u_n - x)^2 + v_n^2} + \int_{u_n}^{t+\epsilon} \frac{(u_n - x)(\varphi^+)dx}{(u_n - x)^2 + v_n^2} + \int_{t+\epsilon}^{b+\epsilon} \frac{(u_n - x)(1)dx}{(u_n - x)^2 + v_n^2},$$

for all n sufficiently large. Proceed as before to get $R_n \geq (\varphi^+ - \varphi^-) \log(v_n) + O(1)$. Combining both inequalities gives $|R_n| \leq -(\varphi^+ - \varphi^-) \log(v_n) + O(1)$ for all n sufficiently large, and (1b) then easily follows.

Consider (2). Now, letting $\varphi^+ := \sup_{x \in (t-\epsilon, t)} \varphi(x) < 1$ and $\varphi^- := \inf_{x \in (t-\epsilon, t)} \varphi(x) > 0$, we show that there exists a constant $c > 0$ such that, for all n sufficiently large,

$$(2a) \quad \pi - \frac{\pi}{2}(1 - \varphi^+) > I_n > \frac{1}{4}\varphi^- \text{ when } u_n - t \leq v_n, \text{ and } \pi - \frac{\pi}{2}(1 - \varphi^+) > I_n > \frac{1}{4}\varphi^- \frac{v_n}{u_n - t} \text{ when } u_n - t \geq v_n.$$

$$(2b) \quad v_n e^{|R_n|} < c v_n^{1-\varphi^+} \text{ when } u_n - t \leq v_n, \text{ and } v_n e^{|R_n|} < c v_n (u_n - t)^{-\varphi^+} \text{ when } u_n - t \geq v_n.$$

Also note that $\frac{1}{4} > \frac{1}{4}\varphi^- \frac{v_n}{u_n - t} > 0$ when $u_n - t \geq v_n$, and so $\sin(\frac{1}{4}\varphi^- \frac{v_n}{u_n - t}) > \frac{1}{8}\varphi^- \frac{v_n}{u_n - t}$. Thus, since $u_n \rightarrow t$ and $v_n \searrow 0$ as $n \rightarrow \infty$, equations (13) and (14) give $(\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n)) \rightarrow (t, 1)$, as required.

Consider (2a). Recall that $\varphi(x) \in [0, 1]$ for all $x \in [a, b]$, $\varphi^+ = \sup_{x \in (t-\epsilon, t)} \varphi(x) < 1$, $\varphi^- = \inf_{x \in (t-\epsilon, t)} \varphi(x) > 0$, $\varphi(x) = 0$ for all $x \in (t, t + \epsilon)$, and $\varphi(x) = 0$ for all $x \in \mathbb{R} \setminus [a, b]$. Equation (12) then gives, for all n ,

$$\begin{aligned} I_n &\leq \int_{a-\epsilon}^{t-\epsilon} \frac{v_n(1)dx}{(u_n - x)^2 + v_n^2} + \int_{t-\epsilon}^t \frac{v_n(\varphi^+)dx}{(u_n - x)^2 + v_n^2} + \int_{t+\epsilon}^{b+\epsilon} \frac{v_n(1)dx}{(u_n - x)^2 + v_n^2}, \\ I_n &\geq \int_{t-\epsilon}^t \frac{v_n(\varphi^-)dx}{(u_n - x)^2 + v_n^2}. \end{aligned}$$

Then, since $u_n \rightarrow t \in [a, b]$ and $v_n \searrow 0$ as $n \rightarrow \infty$, we can proceed as in (1a) to get,

$$\frac{\pi}{2}\varphi^+ - \varphi^+ \arctan\left(\frac{u_n - t}{v_n}\right) + O(v_n) \geq I_n \geq \frac{\pi}{2}\varphi^- - \varphi^- \arctan\left(\frac{u_n - t}{v_n}\right) + O(v_n),$$

for all n . Recall that $\arctan : \mathbb{R} \rightarrow \mathbb{R}$ is strictly increasing with $\arctan(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for all $x \in \mathbb{R}$ and $\arctan(x) < \frac{\pi}{2} - \frac{1}{2x}$ for all $x \geq 1$. Therefore,

$$\begin{aligned} \pi\varphi^+ + O(v_n) &\geq I_n \geq \varphi^- \left(\frac{\pi}{2} - \arctan(1)\right) + O(v_n) && \text{when } u_n - t \leq v_n, \\ \pi\varphi^+ + O(v_n) &\geq I_n \geq \frac{1}{2}\varphi^- \frac{v_n}{u_n - t} + O(v_n) && \text{when } u_n - t \geq v_n, \end{aligned}$$

(2a) follows since $\varphi^+ < 1$ and $\varphi^- > 0$, and since $u_n \rightarrow t$ and $v_n \searrow 0$ as $n \rightarrow \infty$.

Consider (2b). Recall that $\varphi(x) \in [0, 1]$ for all $x \in [a, b]$, $\varphi^+ = \sup_{x \in (t-\epsilon, t)} \varphi(x) < 1$, $\varphi^- = \inf_{x \in (t-\epsilon, t)} \varphi(x) > 0$, $\varphi(x) = 0$ for all $x \in (t, t+\epsilon)$, and $\varphi(x) = 0$ for all $x \in \mathbb{R} \setminus [a, b]$. Choose n sufficiently large that $u_n \in (t - \epsilon, t + \epsilon)$. Then, when $u_n \leq t$, equation (12) gives,

$$\begin{aligned} R_n &\leq \int_{a-\epsilon}^{t-\epsilon} \frac{(u_n - x)(1)dx}{(u_n - x)^2 + v_n^2} + \int_{t-\epsilon}^{u_n} \frac{(u_n - x)(\varphi^+)dx}{(u_n - x)^2 + v_n^2} + \int_{u_n}^t \frac{(u_n - x)(\varphi^-)dx}{(u_n - x)^2 + v_n^2}, \\ R_n &\geq \int_{t-\epsilon}^{u_n} \frac{(u_n - x)(\varphi^-)dx}{(u_n - x)^2 + v_n^2} + \int_{u_n}^t \frac{(u_n - x)(\varphi^+)dx}{(u_n - x)^2 + v_n^2} + \int_{t+\epsilon}^{b+\epsilon} \frac{(u_n - x)(1)dx}{(u_n - x)^2 + v_n^2}. \end{aligned}$$

Also, when $u_n \geq t$,

$$\begin{aligned} R_n &\leq \int_{a-\epsilon}^{t-\epsilon} \frac{(u_n - x)(1)dx}{(u_n - x)^2 + v_n^2} + \int_{t-\epsilon}^t \frac{(u_n - x)(\varphi^+)dx}{(u_n - x)^2 + v_n^2}, \\ R_n &\geq \int_{t-\epsilon}^t \frac{(u_n - x)(\varphi^-)dx}{(u_n - x)^2 + v_n^2} + \int_{t+\epsilon}^{b+\epsilon} \frac{(u_n - x)(1)dx}{(u_n - x)^2 + v_n^2}. \end{aligned}$$

Then, since $u_n \rightarrow t \in [a, b]$ and $v_n \searrow 0$ as $n \rightarrow \infty$, we can proceed as in (1b) to get,

$$\begin{aligned} |R_n| &\leq -(\varphi^+ - \varphi^-) \log(v_n) - \frac{1}{2}\varphi^- \log((u_n - t)^2 + v_n^2) + O(1) && \text{when } u_n \leq t, \\ |R_n| &\leq -\frac{1}{2}\varphi^+ \log((u_n - t)^2 + v_n^2) + O(1) && \text{when } u_n \geq t. \end{aligned}$$

Finally note that $\frac{1}{2} \log((u_n - t)^2 + v_n^2) \geq \log(v_n)$ and $\frac{1}{2} \log((u_n - t)^2 + v_n^2) \geq \log|u_n - t|$. Therefore, for all n sufficiently large, $|R_n| \leq -\varphi^+ \log(v_n) + O(1)$ when $u_n - t \leq v_n$, and $|R_n| \leq -\varphi^+ \log(u_n - t) + O(1)$ when $u_n - t \geq v_n$. (2b) then easily follows. \square

We end this section with the following trivial result:

LEMMA 2.5.

- (1) Assume that the measure, μ , of hypothesis 1.1 is of the form $\mu = \sum_k \lambda_{A_i}$, where $\{A_1, A_2, \dots\}$ are mutually disjoint intervals. Then $R = \mathbb{R}$, and lemmas 2.1 and 2.3 completely describe $\partial\mathcal{L}$.

- (2) Assume that μ is such that $\mathbb{R} \setminus R$ is non-empty, and each $t \in \mathbb{R} \setminus R$ satisfies one of the cases (1-5) of lemma 2.4. Then lemmas 2.1, 2.3 and 2.4 completely describe $\partial\mathcal{L}$.

PROOF. The fact that $\mathbb{R} = R$ in part (1) follows from equation (8). Next, recall that corollary 2.2 implies that $\partial\mathcal{L}$ is the set of all (χ, η) for which there exists a $\{w_n\}_{n \geq 1} \subset \mathbb{H}$ with $(\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n)) \rightarrow (\chi, \eta)$ as $n \rightarrow \infty$, and either $|w_n| \rightarrow \infty$ or $w_n \rightarrow t \in \mathbb{R}$ as $n \rightarrow \infty$. Lemma 2.1 covers all sequences with $|w_n| \rightarrow \infty$, and lemma 2.3 covers all sequences with $w_n \rightarrow t \in R$. Finally, in part (2), lemma 2.4 covers all sequences with $w_n \rightarrow t \in \mathbb{R} \setminus R$, since each such t satisfies one of the cases of lemma 2.4 by assumption. The result trivially follows. \square

2.3. The edge, \mathcal{E} . In this section we prove an analogous result for the edge, \mathcal{E} , to theorem 2.1 for the liquid region, \mathcal{L} . Recall in theorem 2.1, we map \mathcal{L} to \mathbb{H} by mapping $(\chi, \eta) \in \mathcal{L}$ to the unique root of $f'_{(\chi, \eta)}$ in \mathbb{H} . This map is a homeomorphism with inverse $w \mapsto (\chi_{\mathcal{L}}(w), \eta_{\mathcal{L}}(w))$. Recall also (see remark 2.1), that $(\chi_{\mathcal{E}}(\cdot), \eta_{\mathcal{E}}(\cdot)) : R \rightarrow \partial\mathcal{L}$ is the smooth curve, parameterised over the open set $R \subset \mathbb{R}$ (see equations (8) and (11)), which is the unique continuous extensions to R of $(\chi_{\mathcal{L}}(\cdot), \eta_{\mathcal{L}}(\cdot)) : \mathbb{H} \rightarrow \mathcal{L}$. The main result of this section, theorem 2.3, uses definition 1.5 for \mathcal{E} to define a map from \mathcal{E} to \mathbb{R} which is analogous to the map from \mathcal{L} to \mathbb{H} discussed above. This is shown to bijectively map \mathcal{E} to R with inverse $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$. Then, in corollary 2.4, we use theorem 2.3 to prove the equivalence of definitions 1.4 and 1.5 for \mathcal{E} . Lemma 2.6 further explores this equivalence.

Again we denote $f'_{(\chi, \eta)}$ simply by f' where no confusion is possible. Recall definition 1.5: \mathcal{E} is the disjoint union, $\mathcal{E} := \mathcal{E}_{\mu} \cup \mathcal{E}_{\lambda-\mu} \cup \mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2$, where

- \mathcal{E}_{μ} is the set of all $(\chi, \eta) \in [a, b] \times [0, 1]$ with $b \geq \chi \geq \chi + \eta - 1 \geq a$ for which f' has a repeated root in $\mathbb{R} \setminus [\chi + \eta - 1, \chi]$.
- $\mathcal{E}_{\lambda-\mu}$ is the set of all (χ, η) for which f' has a repeated root in $(\chi + \eta - 1, \chi)$.
- \mathcal{E}_0 is the set of all (χ, η) for which $\eta = 1$ and f' has a root at χ ($= \chi + \eta - 1$).
- \mathcal{E}_1 is the set of all (χ, η) for which $\eta < 1$ and f' has a root at χ .
- \mathcal{E}_2 is the set of all (χ, η) for which $\eta < 1$ and f' has a root at $\chi + \eta - 1$.

The fact that the above union is disjoint follows from corollary 3.2 of theorem 3.1. Also, theorem 3.1 implies that f' has at most one real-valued repeated root.

THEOREM 2.3. Define $W_{\mathcal{E}} : \mathcal{E} \rightarrow \mathbb{R}$ by mapping to the corresponding root of f' :

- $W_{\mathcal{E}}(\chi, \eta)$ is the unique real-valued repeated root whenever $(\chi, \eta) \in \mathcal{E}_{\mu} \cup \mathcal{E}_{\lambda-\mu}$.
- $W_{\mathcal{E}}(\chi, \eta)$ is the root χ whenever $(\chi, \eta) \in \mathcal{E}_0 \cup \mathcal{E}_1$.
- $W_{\mathcal{E}}(\chi, \eta)$ is the root $\chi + \eta - 1$ whenever $(\chi, \eta) \in \mathcal{E}_2$.

Then $W_{\mathcal{E}}(\mathcal{E}) = R$ and $W_{\mathcal{E}} : \mathcal{E} \rightarrow R$ is a bijection with inverse $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$.

PROOF. Let $C : \mathbb{C} \setminus \text{Supp}(\mu) \rightarrow \mathbb{C}$ be the analytic extension of the Cauchy transform defined in part (a) of lemma 2.2. Also define R_1 and R_2 as in equation (11). We show:

- (1) $W_{\mathcal{E}}(\mathcal{E}_{\mu}) = R_{\mu} := \{t \in \mathbb{R} \setminus \text{Supp}(\mu) : C(t) \neq 0\}$ and $W_{\mathcal{E}} : \mathcal{E}_{\mu} \rightarrow R_{\mu}$ is a bijection with inverse $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$.

- (2) $W_{\mathcal{E}}(\mathcal{E}_{\lambda-\mu}) = R_{\lambda-\mu} := \mathbb{R} \setminus \text{Supp}(\lambda - \mu)$ and $W_{\mathcal{E}} : \mathcal{E}_{\lambda-\mu} \rightarrow R_{\lambda-\mu}$ is a bijection with inverse $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$.
- (3) $W_{\mathcal{E}}(\mathcal{E}_0) = R_0 := \{t \in \mathbb{R} \setminus \text{Supp}(\mu) : C(t) = 0\}$ and $W_{\mathcal{E}} : \mathcal{E}_0 \rightarrow R_0$ is a bijection with inverse $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$.
- (4) $W_{\mathcal{E}}(\mathcal{E}_1) = R_1$ and $W_{\mathcal{E}} : \mathcal{E}_1 \rightarrow R_1$ is a bijection with inverse $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$.
- (5) $W_{\mathcal{E}}(\mathcal{E}_2) = R_2$ and $W_{\mathcal{E}} : \mathcal{E}_2 \rightarrow R_2$ is a bijection with inverse $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$.

Note, equation (11) implies that R is the disjoint union, $R = R_{\mu} \cup R_{\lambda-\mu} \cup R_0 \cup R_1 \cup R_2$. Parts (1-5) thus easily give the required result. In fact they are a stronger statement.

Consider (1). We prove this by showing:

- (1a) Fix $(\chi, \eta) \in \mathcal{E}_{\mu}$ and let $t := W_{\mathcal{E}}(\chi, \eta)$. Then $t \in R_{\mu}$ and $(\chi, \eta) = (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$.
- (1b) Fix $t \in R_{\mu}$ and let $(\chi, \eta) := (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$. Then $(\chi, \eta) \in \mathcal{E}_{\mu}$ and $W_{\mathcal{E}}(\chi, \eta) = t$.

Consider (1a). Note, equation (7) and the definition of \mathcal{E}_{μ} imply that $t \in (\mathbb{R} \setminus [\chi + \eta - 1, \chi]) \setminus \text{Supp}(\mu)$. Also note, equations (7) and (10) give,

$$(15) \quad f'(w) = C(w) + \log(w - \chi) - \log(w - \chi - \eta + 1),$$

for all $w \in \mathbb{C} \setminus \mathbb{R}$, where \log is principal value. This has a trivial analytic extension to $(\mathbb{C} \setminus [\chi + \eta - 1, \chi]) \setminus \text{Supp}(\mu)$. Also, parts (a) and (b) of corollary 3.2 imply that $(\chi, \eta) \in (a, b) \times (0, 1)$ and $b > \chi > \chi + \eta - 1 > a$. Finally, since t is a repeated root of f' , $e^{f'(t)} = 1$ and $f''(t) = 0$. Therefore,

$$(16) \quad e^{C(t)} \left(\frac{t - \chi}{t - \chi - \eta + 1} \right) = 1 \quad \text{and} \quad C'(t) + \frac{1}{t - \chi} - \frac{1}{t - \chi - \eta + 1} = 0.$$

The first part gives $C(t) \neq 0$, since $\eta < 1$, and so $t \in R_{\mu}$. Also, the first part gives,

$$\frac{1}{t - \chi} = \frac{e^{C(t)} - 1}{1 - \eta} \quad \text{and} \quad \frac{1}{t - \chi - \eta + 1} = \frac{e^{C(t)} - 1}{(1 - \eta)e^{C(t)}}.$$

Substitute into the second part and solve to get $(\chi, \eta) = (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ (see lemma 2.3), as required.

Consider (1b). First note, lemma 2.3 implies that $(\chi, \eta) \in \partial\mathcal{L}$, and so $(\chi, \eta) \in [a, b] \times [0, 1]$ with $b \geq \chi \geq \chi + \eta - 1 \geq a$. Also, this lemma gives,

$$(17) \quad \eta = 1 + \frac{(e^{C(t)} - 1)^2}{e^{C(t)}C'(t)}, \quad \chi = t + \frac{e^{C(t)} - 1}{e^{C(t)}C'(t)}, \quad \chi + \eta - 1 = t + \frac{e^{C(t)} - 1}{C'(t)}.$$

Note that $e^{C(t)} \neq 1$ ($C(t) \neq 0$ since $t \in R_{\mu}$), $e^{C(t)} > 0$ and $C'(t) < 0$ (see part (a) of lemma (2.2)). Thus the second term on the right hand side of the expression for η is negative, and so $\eta < 1$. Also, the second terms on the right hand side of the expressions for χ and $\chi + \eta - 1$ are non-zero and have the same sign, and so $t \in \mathbb{R} \setminus [\chi + \eta - 1, \chi]$. Note, equation (15) holds as above, for all $w \in (\mathbb{C} \setminus [\chi + \eta - 1, \chi]) \setminus \text{Supp}(\mu)$. Substitute the above expressions for χ and η into f' to get $f'(t) = 0$. Similarly $f''(t) = 0$. Thus $(\chi, \eta) \in \mathcal{E}_{\mu}$ by definition, and $W_{\mathcal{E}}(\chi, \eta) = t$, as required.

Consider (2). We prove this by showing:

- (2a) Fix $(\chi, \eta) \in \mathcal{E}_{\lambda-\mu}$ and let $t := W_{\mathcal{E}}(\chi, \eta)$. Then $t \in R_{\lambda-\mu}$ and $(\chi, \eta) = (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$.

(2b) Fix $t \in R_{\lambda-\mu}$ and let $(\chi, \eta) := (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$. Then $(\chi, \eta) \in \mathcal{E}_{\lambda-\mu}$ and $W_{\mathcal{E}}(\chi, \eta) = t$.

Consider (2a). Note, equation (7) and the definition of $\mathcal{E}_{\lambda-\mu}$ give $t \in (\chi + \eta - 1, \chi) \setminus \text{Supp}(\lambda - \mu) \subset R_{\lambda-\mu}$, and so it remains to show that $(\chi, \eta) = (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$. Fix $I = (t_2, t_1) \subset (\chi + \eta - 1, \chi) \setminus \text{Supp}(\lambda - \mu)$ with $t \in I$. Equations (7) and (10) then give,

$$(18) \quad f'(w) = C_I(w) + \log(w - \chi) - \log(w - t_1) + \log(w - t_2) - \log(w - \chi - \eta + 1),$$

for all $w \in \mathbb{C} \setminus \mathbb{R}$, where \log is principal value and $C_I(w) := \int_{[a,b] \setminus I} \frac{\mu[dx]}{w-x}$. This has a trivial analytic extension to $(\mathbb{C} \setminus \mathbb{R}) \cup I$. Also, since t is a repeated root of f' , $e^{f'(t)} = 1$ and $f''(t) = 0$. Equation (16) again holds, where now $e^{C(t)}$ and $C'(t)$ are defined by the analytic extensions of part (b) of lemma (2.2). We can then proceed as for \mathcal{E}_{μ} to get $(\chi, \eta) = (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$, as required.

Consider (2b). First note, lemma 2.3 implies that $(\chi, \eta) \in \partial\mathcal{L}$, and so $(\chi, \eta) \in [a, b] \times [0, 1]$ with $b \geq \chi \geq \chi + \eta - 1 \geq a$. Equation (17) again holds, where now $e^{C(t)}$ and $C'(t)$ are defined by the analytic extensions of part (b) of lemma (2.2). Therefore $e^{C(t)} < 0$ and $C'(t) > 0$. Thus the second term on the right hand side of the expression for η is negative, and so $\eta < 1$. Also the second term on the right hand side of the expression for χ is positive, and the second term on the right hand side of the expression for $\chi + \eta - 1$ is negative, and so $t \in (\chi + \eta - 1, \chi)$. Therefore, fixing $I = (t_2, t_1) \subset (\chi + \eta - 1, \chi) \setminus \text{Supp}(\lambda - \mu)$ with $t \in I$, equation (15) holds as above, for all $w \in (\mathbb{C} \setminus \mathbb{R}) \cup I$. This easily gives $f'(t) = f''(t) = 0$. Thus $(\chi, \eta) \in \mathcal{E}_{\lambda-\mu}$ by definition, and $W_{\mathcal{E}}(\chi, \eta) = t$, as required.

Consider (3). We prove this by showing:

(3a) Fix $(\chi, \eta) \in \mathcal{E}_0$ and let $t := W_{\mathcal{E}}(\chi, \eta)$. Then $t \in R_0$ and $(\chi, \eta) = (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$.

(3b) Fix $t \in R_0$ and let $(\chi, \eta) := (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$. Then $(\chi, \eta) \in \mathcal{E}_0$ and $W_{\mathcal{E}}(\chi, \eta) = t$.

Consider (3a). The definitions of \mathcal{E}_0 and $W_{\mathcal{E}}$ give $(\chi, \eta) = (t, 1)$. Moreover f' has a root at $t = \chi$, and so equation (7) gives $t \in \mathbb{R} \setminus \text{Supp}(\mu)$. Also equations (7) and (10) give $f'(w) = C(w)$ for all $w \in \mathbb{C} \setminus \mathbb{R}$. This has a trivial analytic extension to $\mathbb{C} \setminus \text{Supp}(\mu)$, and so $f'(t) = C(t)$. Thus, since f' has a root at $t = \chi$, $C(t) = 0$ and lemma 2.3 gives $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) = (t, 1)$. Thus $t \in R_0$ and $(\chi, \eta) = (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$, as required. Consider (3b). The definition of R_0 gives $t \in \mathbb{R} \setminus \text{Supp}(\mu)$ and $C(t) = 0$. Lemma 2.3 thus gives $(\chi, \eta) = (t, 1)$. Then, as before, $f'(w) = C(w)$ for all $w \in \mathbb{C} \setminus \text{Supp}(\mu)$, and so $f'(t) = C(t) = 0$. Therefore $(\chi, \eta) \in \mathcal{E}_0$ by definition, and $W_{\mathcal{E}}(\chi, \eta) = t$, as required.

Consider (4). We prove this by showing:

(4a) Fix $(\chi, \eta) \in \mathcal{E}_1$ and let $t := W_{\mathcal{E}}(\chi, \eta)$. Then $t \in R_1$ and $(\chi, \eta) = (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$.

(4b) Fix $t \in R_1$ and let $(\chi, \eta) := (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$. Then $(\chi, \eta) \in \mathcal{E}_1$ and $W_{\mathcal{E}}(\chi, \eta) = t$.

Consider (4a). Note, the definitions of \mathcal{E}_1 and $W_{\mathcal{E}}$ give $\eta < 1$ and $t = \chi$. Moreover, f' has a root at $t = \chi$, and so equation (7) shows that there exists an interval, $I = (t_2, t_1)$, with $t = \chi \in I$, $(t, t_1) \subset \mathbb{R} \setminus \text{Supp}(\mu)$ and $(t_2, t) \subset (\chi + \eta - 1, \chi) \setminus \text{Supp}(\lambda - \mu)$. Equation (11) then implies that $t \in R_1$. Moreover, equations (7) and (10) give,

$$(19) \quad f'(w) = C_I(w) + \log(w - t_2) - \log(w - \chi - \eta + 1),$$

for all $w \in \mathbb{C} \setminus \mathbb{R}$, where \log is principal value. This has a trivial analytic extension to $(\mathbb{C} \setminus \mathbb{R}) \cup I$. Thus, since $t = \chi$ and $f'(\chi) = 0$, $C_I(t) + \log(t - t_2) - \log(1 - \eta) = 0$, where now \log denotes natural logarithm. Solving gives $\eta = 1 - (t - t_2)e^{C_I(t)}$. Lemma (2.3) then gives $(\chi, \eta) = (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$, as required.

Consider (4b). Note, lemma 2.3 implies that $(\chi, \eta) \in \partial\mathcal{L}$, and so $(\chi, \eta) \in [a, b] \times [0, 1]$ with $b \geq \chi \geq \chi + \eta - 1 \geq a$. Also, this lemma gives $\chi = t$ and $\eta = 1 - e^{C_I(t)}(t - t_2) < 1$. Also, since $t \in R_1$, equation (11) implies that we can fix $I = (t_2, t_1)$ with $t = \chi \in I$, $(t, t_1) \subset \mathbb{R} \setminus \text{Supp}(\mu)$ and $(t_2, t) \subset (\chi + \eta - 1, \chi) \setminus \text{Supp}(\lambda - \mu)$. Therefore equation (19) holds as above, for all $w \in (\mathbb{C} \setminus \mathbb{R}) \cup I$. This easily gives $f'(t) = f'(\chi) = 0$. Thus $(\chi, \eta) \in \mathcal{E}_1$ by definition, and $W_{\mathcal{E}}(\chi, \eta) = t$, as required.

Consider (5). We prove this by showing:

(5a) Fix $(\chi, \eta) \in \mathcal{E}_2$ and let $t := W_{\mathcal{E}}(\chi, \eta)$. Then $t \in R_2$ and $(\chi, \eta) = (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$.

(5b) Fix $t \in R_2$ and let $(\chi, \eta) := (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$. Then $(\chi, \eta) \in \mathcal{E}_2$ and $W_{\mathcal{E}}(\chi, \eta) = t$.

Consider (5a). Note, the definitions of \mathcal{E}_2 and $W_{\mathcal{E}}$ give $\eta < 1$ and $t = \chi + \eta - 1$. Moreover, f' has a root at $t = \chi + \eta - 1$, and so equation (7) shows that there exists an interval, $I = (t_2, t_1)$, with $t = \chi + \eta - 1 \in I$, $(t, t_1) \subset (\chi + \eta - 1, \chi) \setminus \text{Supp}(\lambda - \mu)$ and $(t_2, t) \subset \mathbb{R} \setminus \text{Supp}(\mu)$. Equation (11) then implies that $t \in R_2$. Moreover, equations (7) and (10) give,

$$(20) \quad f'(w) = C_I(w) + \log(w - \chi) - \log(w - t_1),$$

for all $w \in \mathbb{C} \setminus \mathbb{R}$, where \log is principal value. This has a trivial analytic extension to $(\mathbb{C} \setminus \mathbb{R}) \cup I$. Thus, since $t = \chi + \eta - 1$ and $f'(\chi + \eta - 1) = 0$, $C_I(t) + \log(1 - \eta) - \log(t_1 - t) = 0$, where now \log denotes natural logarithm. Solving gives $\eta = 1 - (t_1 - t)e^{-C_I(t)}$, and so $\chi = t + 1 - \eta = t + (t_1 - t)e^{-C_I(t)}$. Lemma (2.3) then gives $(\chi, \eta) = (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$, as required. Finally, (5b) follows in a similar way to (4b). \square

Theorem 2.3 only uses definition 1.5 for \mathcal{E} , and shows that $(\chi_{\mathcal{E}}(\cdot), \eta_{\mathcal{E}}(\cdot)) : R \rightarrow \partial\mathcal{L}$ bijectively maps R to \mathcal{E} . Definition 1.4 defines \mathcal{E} as the image space of this map. Therefore:

COROLLARY 2.4. *Definitions 1.4 and 1.5 of the edge, \mathcal{E} , are equivalent.*

We end this section with a result which further clarifies the above equivalence:

LEMMA 2.6. *Define $R_{\mu}, R_{\lambda-\mu}, R_0, R_1, R_2$ as in parts (1-5) in the proof of theorem 2.3. Also, for all $t \in R = R_{\mu} \cup R_{\lambda-\mu} \cup R_0 \cup R_1 \cup R_2$, define the function,*

$$f'_t := f'_{(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))},$$

where the right hand side is defined in equation (7). Then,

- (a) $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) \in \mathcal{E}_{\mu}$ if and only if $t \in R_{\mu}$. Moreover, in this case, t is a root of f'_t of multiplicity either 2 or 3.
- (b) $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) \in \mathcal{E}_{\lambda-\mu}$ if and only if $t \in R_{\lambda-\mu}$. Moreover, in this case, t is a root of f'_t of multiplicity either 2 or 3.
- (c) $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) \in \mathcal{E}_0$ if and only if $t \in R_0$. Moreover, in this case, the functions f'_t and C are equal, and t is a root of f'_t of multiplicity 1.

- (d) $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) \in \mathcal{E}_1$ if and only if $t \in R_1$. Moreover, in this case, t is a root of f'_t of multiplicity either 1 or 2.
- (e) $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) \in \mathcal{E}_2$ if and only if $t \in R_2$. Moreover, in this case, t is a root of f'_t of multiplicity either 1 or 2.

PROOF. Consider (a). The fact that $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) \in \mathcal{E}_{\mu}$ if and only if $t \in R_{\mu}$ follows from part (1) of theorem 2.3. Also, fixing $t \in R_{\mu}$ and defining $(\chi, \eta) := (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) \in \mathcal{E}_{\mu}$, theorem 2.3 gives $t = W_{\mathcal{E}}(\chi, \eta)$. The definitions of \mathcal{E}_{μ} and $W_{\mathcal{E}}$ then imply that t is a repeated root of f'_t , i.e., t has multiplicity at least 2. Finally, theorem 3.1 implies that t has multiplicity at most 3, as required. Part (b) follows similarly.

Consider (c). The fact that $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) \in \mathcal{E}_0$ if and only if $t \in R_0$ follows from part (3) of theorem 2.3. Also, fixing $t \in R_0$ and defining $(\chi, \eta) := (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) \in \mathcal{E}_0$, theorem 2.3 gives $t = W_{\mathcal{E}}(\chi, \eta)$. The definitions of \mathcal{E}_0 and $W_{\mathcal{E}}$ then imply that $\eta = 1$ and t is a root of f'_t . Also, since $\eta = 1$, equations (7) and (10) give $f'_t = C$. Finally, $f''_t(t) = C'(t) = -\int_a^b \frac{\mu[dx]}{(t-x)^2} < 0$, and so t is a root of multiplicity 1, as required.

Consider (d). The fact that $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) \in \mathcal{E}_1$ if and only if $t \in R_1$ follows from part (4) of theorem 2.3. Also, fixing $t \in R_1$ and defining $(\chi, \eta) := (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) \in \mathcal{E}_1$, theorem 2.3 gives $t = W_{\mathcal{E}}(\chi, \eta) = \chi$. The definitions of \mathcal{E}_1 and $W_{\mathcal{E}}$ then imply that $t = \chi$ is a root of f'_t . It remains to show that $t = \chi$ has multiplicity at most 2. Note that part (b) of corollary 3.2 implies that $\eta > 0$. Then, using the notation of theorem 3.1, this theorem implies that $t = \chi \in J$ whenever S_2 is non-empty. The result, whenever S_2 is non-empty, then follows from parts (1) and (9) of theorem 3.1. Whenever S_2 is empty, the result follows from parts (11) and (12). This covers all possibilities. Part (e) follows similarly. \square

2.4. Local geometric properties of the edge curve. Recall (see definition 1.4) that \mathcal{E} is the image of the smooth curve defined by $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ for all $t \in R$, where $R = (\mathbb{R} \setminus \text{Supp}(\mu)) \cup (\mathbb{R} \setminus \text{Supp}(\lambda - \mu)) \cup R_1 \cup R_2$, and R_1, R_2 are defined in equation (11). In this section we investigate the local geometric properties of the edge curve. We show that the curve behaves locally either like a parabola with negative curvature, or a cusp of first order, and this behaviour is completely determined by the multiplicities of the roots as presented in lemma 2.6.

We begin by decomposing the image curve into orthogonal components and performing a Taylor expansion:

LEMMA 2.7. *Fix $t \in R$ and define the (un-normalised) orthogonal vectors, $\mathbf{x} = \mathbf{x}(t)$ and $\mathbf{y} = \mathbf{y}(t)$ as,*

- $\mathbf{x} := (1, e^{C(t)} - 1)$ and $\mathbf{y} := (e^{C(t)} - 1, -1)$ when $t \in (\mathbb{R} \setminus \text{Supp}(\mu)) \cup (\mathbb{R} \setminus \text{Supp}(\lambda - \mu))$.
- $\mathbf{x} := (0, 1)$ and $\mathbf{y} := (1, 0)$ when $t \in R_1$.
- $\mathbf{x} := (1, -1)$ and $\mathbf{y} := (1, 1)$ when $t \in R_2$.

Then,

$$(21) \quad (\chi_{\mathcal{E}}(s), \eta_{\mathcal{E}}(s)) - (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) = a(s)\mathbf{x} + b(s)\mathbf{y},$$

for all $s \in R$ sufficiently close to t , where

$$\begin{aligned} a(s) &= a_1(s-t) + a_2(s-t)^2 + O((s-t)^3), \\ b(s) &= b_1(s-t)^2 + b_2(s-t)^3 + O((s-t)^4), \end{aligned}$$

and $a_1 = a_1(t)$, $a_2 = a_2(t)$, $b_1 = b_1(t)$ and $b_2 = b_2(t)$ are defined in the proof. (Above, when $t \in (\mathbb{R} \setminus \text{Supp}(\mu)) \cup (\mathbb{R} \setminus \text{Supp}(\lambda - \mu))$, $e^{C(t)}$ is defined by the analytic extensions of parts (a) and (b) of lemma 2.2.)

PROOF. First suppose that $t \in (\mathbb{R} \setminus \text{Supp}(\mu)) \cup (\mathbb{R} \setminus \text{Supp}(\lambda - \mu))$. Fix an interval, $I := (t_2, t_1)$, with $t \in I$, $I \subset \mathbb{R} \setminus \text{Supp}(\mu)$ whenever $t \in \mathbb{R} \setminus \text{Supp}(\mu)$, and $I \subset \mathbb{R} \setminus \text{Supp}(\lambda - \mu)$ whenever $t \in \mathbb{R} \setminus \text{Supp}(\lambda - \mu)$. Then, solving equation (21) gives,

$$\begin{aligned} (1 + (e^{C(t)} - 1)^2)a(s) &= (\chi_{\mathcal{E}}(s) - \chi_{\mathcal{E}}(t)) + (\eta_{\mathcal{E}}(s) - \eta_{\mathcal{E}}(t))(e^{C(t)} - 1), \\ (1 + (e^{C(t)} - 1)^2)b(s) &= (\chi_{\mathcal{E}}(s) - \chi_{\mathcal{E}}(t))(e^{C(t)} - 1) - (\eta_{\mathcal{E}}(s) - \eta_{\mathcal{E}}(t)), \end{aligned}$$

for all $s \in I$. Also, lemma 2.3 and parts (a) and (b) of lemma 2.2 give

$$(22) \quad \chi'_{\mathcal{E}}(t) = -\frac{C''(t)(e^{C(t)} - 1) - C'(t)^2(e^{C(t)} + 1)}{e^{C(t)}C'(t)^2} \quad \text{and} \quad \eta'_{\mathcal{E}}(t) = (e^{C(t)} - 1)\chi'_{\mathcal{E}}(t).$$

The second part of this equation and Taylor expansions give the required result with,

- $a_1 = \chi'_{\mathcal{E}}(t)$.
- $2a_2 = \chi''_{\mathcal{E}}(t) + (1 + (e^{C(t)} - 1)^2)^{-1}e^{C(t)}(e^{C(t)} - 1)C'(t)\chi'_{\mathcal{E}}(t)$.
- $2b_1 = -(1 + (e^{C(t)} - 1)^2)^{-1}e^{C(t)}C'(t)\chi'_{\mathcal{E}}(t)$.
- $6b_2 = -(1 + (e^{C(t)} - 1)^2)^{-1}e^{C(t)}[2C''(t)\chi''_{\mathcal{E}}(t) + (C'(t)^2 + C'''(t))\chi'_{\mathcal{E}}(t)]$.

Next suppose that $t \in R_1$. Fix an interval, $I = (t_2, t_1)$, as in equation (11), i.e., with $t \in I$, $(t, t_1) \subset \mathbb{R} \setminus \text{Supp}(\mu)$ and $(t_2, t) \subset \mathbb{R} \setminus \text{Supp}(\lambda - \mu)$. Then, solving equation (21),

$$a(s) = \eta_{\mathcal{E}}(s) - \eta_{\mathcal{E}}(t) \quad \text{and} \quad b(s) = \chi_{\mathcal{E}}(s) - \chi_{\mathcal{E}}(t),$$

for all $s \in I$. Also, part (c) of lemma 2.2 gives,

$$e^{-C(s)} = e^{-C_I(s)} \left(\frac{s-t}{s-t_2} \right),$$

for all $s \in I$, where $C_I(s) := \int_{[a,b] \setminus I} \frac{\mu[dx]}{s-x}$. Lemma 2.3 then gives,

$$\eta_{\mathcal{E}}(s) = 1 - (t - t_2)e^{C_I(s)} + (s - t)h_1(s) \quad \text{and} \quad \chi_{\mathcal{E}}(s) = t + (s - t)^2h_2(s),$$

for all $s \in I$, where

$$h_1(s) := \frac{(t - t_2)(s - t_2)e^{2C_I(s)}C'_I(s) + (s + t - 2t_2)e^{2C_I(s)} - 2(s - t_2)e^{C_I(s)} + (s - t)}{-(t - t_2)e^{C_I(s)} + (s - t)(s - t_2)e^{C_I(s)}C'_I(s)},$$

$$h_2(s) := \frac{(s - t_2)e^{C_I(s)}C'_I(s) + e^{C_I(s)} - 1}{-(t - t_2)e^{C_I(s)} + (s - t)(s - t_2)e^{C_I(s)}C'_I(s)}.$$

Taylor expansions then give the required result with,

- $a_1 = -(t - t_2)e^{C_I(t)}C'_I(t) + h_1(t)$.

- $2a_2 = -(t - t_2)e^{C_I(t)}C_I'(t)^2 - (t - t_2)e^{C_I(t)}C_I''(t) + 2h_1'(t).$
- $b_1 = h_2(t).$
- $b_2 = h_2'(t).$

Next suppose that $t \in R_2$. Fix an interval, $I = (t_2, t_1)$, as in equation (11), i.e., with $t \in I$, $(t, t_1) \subset \mathbb{R} \setminus \text{Supp}(\lambda - \mu)$ and $(t_2, t) \subset \mathbb{R} \setminus \text{Supp}(\mu)$. Then, solving equation (21),

$$\begin{aligned} 2a(s) &= (\chi_{\mathcal{E}}(s) - \eta_{\mathcal{E}}(s)) - (\chi_{\mathcal{E}}(t) - \eta_{\mathcal{E}}(t)), \\ 2b(s) &= (\chi_{\mathcal{E}}(s) + \eta_{\mathcal{E}}(s)) - (\chi_{\mathcal{E}}(t) + \eta_{\mathcal{E}}(t)), \end{aligned}$$

for all $s \in I$. Also part (d) of lemma 2.2 gives,

$$e^{C(s)} = e^{C_I(s)} \left(\frac{s - t}{s - t_1} \right),$$

for all $s \in I$. Lemma 2.3 then gives,

$$\begin{aligned} \chi_{\mathcal{E}}(s) - \eta_{\mathcal{E}}(s) &= t - 1 - 2(t - t_1)e^{-C_I(s)} + (s - t)h_3(s), \\ \chi_{\mathcal{E}}(s) + \eta_{\mathcal{E}}(s) &= t + 1 + (s - t)^2h_4(s), \end{aligned}$$

for all $s \in I$, where

$$\begin{aligned} h_3(s) &:= 1 + \frac{2(t - t_1)(s - t_1)e^{-C_I(s)}C_I'(s) - 2(s + t - 2t_1)e^{-C_I(s)} + 3(s - t_1) - (s - t)e^{C_I(s)}}{t - t_1 + (s - t)(s - t_1)C_I'(s)}, \\ h_4(s) &:= \frac{(s - t_1)C_I'(s) + e^{C_I(s)} - 1}{t - t_1 + (s - t)(s - t_1)C_I'(s)}. \end{aligned}$$

Taylor expansions then give the required result with,

- $2a_1 = 2(t - t_1)e^{-C_I(t)}C_I'(t) + h_3(t).$
- $2a_2 = -(t - t_1)e^{-C_I(t)}C_I'(t)^2 + (t - t_1)e^{-C_I(t)}C_I''(t) + h_3'(t).$
- $2b_1 = h_4(t).$
- $2b_2 = h_4'(t).$

□

The local geometric behaviour of the edge curve, as examined in lemma 2.7, depends on the parameters a_1, a_2, b_1, b_2 . The next lemma further clarifies this by classifying the situations in which these terms are zero or non-zero. We show that these situations are completely determined by the multiplicities of the roots as presented in lemma 2.6:

LEMMA 2.8. *Fix $t \in R = R_\mu \cup R_{\lambda-\mu} \cup R_0 \cup R_1 \cup R_2$, and define f_t' as in lemma 2.6. Then the following 9 cases exhaust all possibilities:*

- (1) $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) \in \mathcal{E}_\mu$, where $t \in R_\mu$ is a root of f_t' of multiplicity 2.
- (2) $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) \in \mathcal{E}_\mu$, where $t \in R_\mu$ is a root of f_t' of multiplicity 3.
- (3) $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) \in \mathcal{E}_{\lambda-\mu}$, where $t \in R_{\lambda-\mu}$ is a root of f_t' of multiplicity 2.
- (4) $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) \in \mathcal{E}_{\lambda-\mu}$, where $t \in R_{\lambda-\mu}$ is a root of f_t' of multiplicity 3.
- (5) $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) \in \mathcal{E}_0$, where $t \in R_0$ is a root of f_t' of multiplicity 1.
- (6) $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) \in \mathcal{E}_1$, where $t \in R_1$ is a root of f_t' of multiplicity 1.
- (7) $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) \in \mathcal{E}_1$, where $t \in R_1$ is a root of f_t' of multiplicity 2.

(8) $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) \in \mathcal{E}_2$, where $t \in R_2$ is a root of f'_t of multiplicity 1.

(9) $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) \in \mathcal{E}_2$, where $t \in R_2$ is a root of f'_t of multiplicity 2.

Moreover, defining $a_1 = a_1(t)$, $a_2 = a_2(t)$, $b_1 = b_1(t)$ and $b_2 = b_2(t)$ as in lemma 2.7,

- $a_1 \neq 0$ and $b_1 \neq 0$ in cases (1, 3, 5, 6, 8).
- $a_1 = b_1 = 0$, $a_2 \neq 0$ and $b_2 \neq 0$ in cases (2, 4, 7, 9).

PROOF. The fact that cases (1-9) exhaust all possibilities follows trivially from lemma 2.6. Next, using the various analytic extensions of lemma 2.2, we will show:

(a) In cases (1, 2, 3, 4), i.e., when $t \in R_{\mu} \cup R_{\lambda-\mu}$ is a root of multiplicity 2 or 3,

$$a_1 = - \left(\frac{e^{C(t)} - 1}{e^{C(t)} C'(t)^2} \right) f_t'''(t) \quad \text{and} \quad b_1 = \frac{1}{2} \left(\frac{e^{C(t)} - 1}{C'(t)(1 + (e^{C(t)} - 1)^2)} \right) f_t'''(t).$$

(b) In cases (2, 4), i.e., when $t \in R_{\mu} \cup R_{\lambda-\mu}$ is a root of multiplicity 3,

$$a_2 = - \frac{1}{2} \left(\frac{e^{C(t)} - 1}{e^{C(t)} C'(t)^2} \right) f_t^{(4)}(t) \quad \text{and} \quad b_2 = \frac{1}{3} \left(\frac{e^{C(t)} - 1}{C'(t)(1 + (e^{C(t)} - 1)^2)} \right) f_t^{(4)}(t).$$

(c) In case (5), i.e., when $t \in R_0$ is a root of multiplicity 1,

$$a_1 = 2 \quad \text{and} \quad b_1 = -C'(t) = -f_t''(t) = \int_a^b \frac{\mu[dx]}{(t-x)^2}.$$

(d) In cases (6, 7), i.e., when $t \in R_1$ is a root of multiplicity 1 or 2,

$$a_1 = -2(t - t_2)e^{C_I(t)} f_t''(t) \quad \text{and} \quad b_1 = -f_t''(t).$$

(e) In case (7), i.e., when $t \in R_1$ is a root of multiplicity 2,

$$a_2 = -\frac{3}{2}(t - t_2)e^{C_I(t)} f_t'''(t) \quad \text{and} \quad b_2 = -f_t'''(t).$$

(f) In cases (8, 9), i.e., when $t \in R_2$ is a root of multiplicity 1 or 2,

$$a_1 = 2(t - t_1)e^{-C_I(t)} f_t''(t) \quad \text{and} \quad b_1 = \frac{1}{2} f_t''(t).$$

(g) In case (9), i.e., when $t \in R_2$ is a root of multiplicity 2,

$$a_2 = \frac{3}{2}(t - t_1)e^{-C_I(t)} f_t'''(t) \quad \text{and} \quad b_2 = \frac{1}{2} f_t'''(t).$$

The result trivially follows from (a-g). For example, in case (1), (a) implies that $a_1 \neq 0$ and $a_2 \neq 0$. Also, in case (2), (a) and (b) imply that $a_1 = b_1 = 0$, $a_2 \neq 0$ and $b_2 \neq 0$, etc.

For simplicity of notation, set $(\chi, \eta) := (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ for the remainder of this lemma. Consider (a) when $t \in R_{\mu}$. Following the proof of part (1) of theorem 2.3, $\eta < 1$, $t \in (\mathbb{R} \setminus [\chi + \eta - 1, \chi]) \setminus \text{Supp}(\mu)$, and equation (15) holds for all $w \in (\mathbb{C} \setminus [\chi + \eta - 1, \chi]) \setminus \text{Supp}(\mu)$. Also equation (17) holds where $C(t)$ and $C'(t)$ are defined by the analytic extensions of part (a) of lemma 2.2. Differentiate equation (15) twice, take $w = t$, and substitute $t - \chi$ and $t - \chi - \eta + 1$ from equation (17) to get,

$$(23) \quad f_t'''(t) = C'''(t) - C'(t)^2 \frac{e^{C(t)} + 1}{e^{C(t)} - 1},$$

when $t \in R_\mu$ (note $e^{C(t)} - 1 \neq 0$, since $C(t) \neq 0$ by definition). Equation (22) then gives,

$$(24) \quad \chi'_\mathcal{E}(t) = - \left(\frac{e^{C(t)} - 1}{e^{C(t)} C'(t)^2} \right) f_t'''(t).$$

Part (a), when $t \in R_\mu$, then follows from the expressions of a_1 and b_1 given in the proof of lemma 2.7.

Consider (a) when $t \in R_{\lambda-\mu}$. Following the proof of part (2) of theorem 2.3, $\eta < 1$ and $t \in (\chi + \eta - 1, \chi) \setminus \text{Supp}(\lambda - \mu)$. Also, fixing $I = (t_2, t_1) \subset (\chi + \eta - 1, \chi) \setminus \text{Supp}(\lambda - \mu)$ with $t \in I$, equation (18) holds for all $w \in (\mathbb{C} \setminus \mathbb{R}) \cup I$. Also, equation (17) holds, where $e^{C(t)}$ and $C'(t)$ are defined by the analytic extensions of part (b) of lemma (2.2). Differentiate equation (18) and use the analytic extension for C' to get,

$$(25) \quad f_t''(w) = C'(w) + \frac{1}{w - \chi} - \frac{1}{w - \chi - \eta + 1},$$

for all $w \in (\mathbb{C} \setminus \mathbb{R}) \cup I$. Differentiate again, take $w = t$, and substitute $t - \chi$ and $t - \chi - \eta + 1$ from equation (17) to show that equation (23) holds in this case also. Finally, we can proceed as above to show part (a) when $t \in R_{\lambda-\mu}$.

Consider (b). First note that $f_t'''(t) = 0$, since t is a root of f_t' of multiplicity 3. Equation (23) thus gives,

$$C'''(t) = C'(t)^2 \frac{e^{C(t)} + 1}{e^{C(t)} - 1}.$$

Next, differentiate the first part of equation (22), and substitute the above expression of $C'''(t)$ to get,

$$\chi''_\mathcal{E}(t) = - \frac{e^{C(t)} - 1}{e^{C(t)} C'(t)^2} \left(C'''(t) - 2C'(t)^3 \frac{e^{2C(t)} + e^{C(t)} + 1}{(e^{C(t)} - 1)^2} \right).$$

Finally, differentiate equation (15) three times (when $t \in R_\mu$) or differentiate equation (25) twice (when $t \in R_{\lambda-\mu}$), take $w = t$, and substitute $t - \chi$ and $t - \chi - \eta + 1$ from equation (17) to get,

$$f_t^{(4)}(t) = C'''(t) - 2C'(t)^3 \frac{e^{2C(t)} + e^{C(t)} + 1}{(e^{C(t)} - 1)^2}.$$

Therefore

$$\chi''_\mathcal{E}(t) = - \left(\frac{e^{C(t)} - 1}{e^{C(t)} C'(t)^2} \right) f_t^{(4)}(t).$$

Also $\chi'_\mathcal{E}(t) = 0$ since $f_t'''(t) = 0$ (see equation (24)). Part (b) then follows from the expressions of a_2 and b_2 given in the proof of lemma 2.7.

Consider (c). Recall that $t \in R_0$, i.e., $t \in \mathbb{R} \setminus \text{Supp}(\mu)$ and $C(t) = 0$. Following the proof of part (3) of theorem 2.3, $(\chi, \eta) = (t, 1)$, and $f_t'(w) = C(w)$ for all $w \in \mathbb{C} \setminus \text{Supp}(\mu)$. Therefore $C(t) = 0$, $f_t'(t) = C(t)$, and equation (10) gives,

$$f_t''(t) = C'(t) = - \int_a^b \frac{\mu[dx]}{(t - x)^2} < 0.$$

Equation (22) then gives $\chi'_\varepsilon(t) = 2$. Part (c) then follows from the expressions of a_1 and b_1 given in the proof of lemma 2.7.

Consider (d). Following the proof of part (4) of theorem 2.3, $\eta < 1$ and $t = \chi$. Also, fixing $I = (t_2, t_1)$ with $t = \chi \in I$, $(t, t_1) \subset \mathbb{R} \setminus \text{Supp}(\mu)$ and $(t_2, t) \subset (\chi + \eta - 1, \chi) \setminus \text{Supp}(\lambda - \mu)$, equation (19) holds for all $w \in (\mathbb{C} \setminus \mathbb{R}) \cup I$. Differentiate this equation, take $w = t$, and substitute $\chi = t$ and $\eta = 1 - e^{C_I(t)}(t - t_2)$ (see lemma 2.3) to get,

$$f_t''(t) = C_I'(t) + \frac{1}{t - t_2} - \frac{1}{(t - t_2)e^{C_I(t)}}.$$

Part (d) then follows from the expressions of a_1 and b_1 given in the proof of lemma 2.7.

Consider (e). First note that $f_t''(t) = 0$, since t is a root of f_t' of multiplicity 2. The above expression of $f_t''(t)$ thus gives,

$$C_I'(t) = \frac{1}{(t - t_2)e^{C_I(t)}} - \frac{1}{t - t_2}.$$

Next, substitute the above expression of $C_I'(t)$ into the expressions of a_2 and b_2 given in the proof of lemma 2.7 to get,

$$\begin{aligned} a_2 &= -\frac{3}{2}(t - t_2)e^{C_I(t)} \left(C_I''(t) - \frac{1}{(t - t_2)^2} + \frac{1}{(t - t_2)^2 e^{2C_I(t)}} \right), \\ b_2 &= - \left(C_I''(t) - \frac{1}{(t - t_2)^2} + \frac{1}{(t - t_2)^2 e^{2C_I(t)}} \right). \end{aligned}$$

Finally, differentiate equation (19) twice, take $w = t$, and substitute $\chi = t$ and $\eta = 1 - e^{C_I(t)}(t - t_2)$ to get,

$$f_t'''(t) = C_I''(t) - \frac{1}{(t - t_2)^2} + \frac{1}{(t - t_2)^2 e^{2C_I(t)}}.$$

Part (e) easily follows.

Consider (f). Following the proof of part (5) of theorem 2.3, $\eta < 1$ and $t = \chi + \eta - 1$. Also, fixing $I = (t_2, t_1)$ with $t = \chi + \eta - 1 \in I$, $(t, t_1) \subset (\chi + \eta - 1, \chi) \setminus \text{Supp}(\lambda - \mu)$ and $(t_2, t) \subset \mathbb{R} \setminus \text{Supp}(\mu)$, equation (20) holds for all $w \in (\mathbb{C} \setminus \mathbb{R}) \cup I$. Differentiate this equation, take $w = t$, and substitute $\chi = t - e^{-C_I(t)}(t - t_1)$ and $\eta = 1 + e^{-C_I(t)}(t - t_1)$ (see lemma 2.3) to get,

$$f_t''(t) = C_I'(t) + \frac{1}{(t - t_1)e^{-C_I(t)}} - \frac{1}{t - t_1}.$$

Part (f) then follows from the expressions of a_1 and b_1 given in the proof of lemma 2.7.

Consider (g). First note that $f_t''(t) = 0$, since t is a root of f_t' of multiplicity 2. The above expression of $f_t''(t)$ thus gives,

$$C_I'(t) = \frac{1}{t - t_1} - \frac{1}{(t - t_1)e^{-C_I(t)}}.$$

Next, substitute the above expression of $C_I'(t)$ into the expressions of a_2 and b_2 given in the proof of lemma 2.7 to get,

$$a_2 = \frac{3}{2}(t - t_1)e^{-C_I(t)} \left(C_I''(t) - \frac{1}{(t - t_1)^2 e^{-2C_I(t)}} + \frac{1}{(t - t_1)^2} \right),$$

$$b_2 = \frac{1}{2} \left(C_I''(t) - \frac{1}{(t - t_1)^2 e^{-2C_I(t)}} + \frac{1}{(t - t_1)^2} \right).$$

Finally, differentiate equation (20) twice, take $w = t$, and substitute $\chi = t - e^{-C_I(t)}(t - t_1)$ and $\eta = 1 + e^{-C_I(t)}(t - t_1)$ to get,

$$f_t'''(t) = C_I''(t) - \frac{1}{(t - t_1)^2 e^{-2C_I(t)}} + \frac{1}{(t - t_1)^2}.$$

Part (g) easily follows. \square

We end this section with a result which clarifies and summaries all possible cases:

LEMMA 2.9. *Fix $t \in R = R_\mu \cup R_{\lambda-\mu} \cup R_0 \cup R_1 \cup R_2$, and define f_t' as in lemma 2.6. Also define, as in lemma 2.7, $\mathbf{x} = \mathbf{x}(t)$, $\mathbf{y} = \mathbf{y}(t)$, $a_1 = a_1(t)$, $a_2 = a_2(t)$, $b_1 = b_1(t)$ and $b_2 = b_2(t)$. Consider the exhaustive cases, (1-9), of lemma 2.8:*

Case (1): *When $t \in R_\mu$ is a root of f_t' of multiplicity 2,*

- $(\chi_\mathcal{E}(t), \eta_\mathcal{E}(t))$, given by lemma 2.3, is in the interior of the shape in the middle of figure 6.
- $(\chi_\mathcal{E}(s), \eta_\mathcal{E}(s)) - (\chi_\mathcal{E}(t), \eta_\mathcal{E}(t)) = a(s)\mathbf{x} + b(s)\mathbf{y}$ for all $s \in R$ close to t .
- $\mathbf{x} = (1, e^{C(t)} - 1)$ and $\mathbf{y} = (e^{C(t)} - 1, -1)$.
- $a(s) = a_1(s - t) + O((s - t)^2)$ and $b(s) = b_1(s - t)^2 + O((s - t)^3)$.
- $a_1 \neq 0$ and $b_1 \neq 0$ are given in part (a) of the proof of lemma 2.8.
- The edge curve behaves like a parabola with negative curvature in a neighbourhood of $(\chi_\mathcal{E}(t), \eta_\mathcal{E}(t))$, with tangent vector \mathbf{x} and normal vector \mathbf{y} .

Case (2): *The set of all $t \in R_\mu$ for which t is a root of f_t' of multiplicity 3 is a discrete subset of (a, b) , i.e., it is composed of isolated singletons. Moreover, in this case,*

- $(\chi_\mathcal{E}(t), \eta_\mathcal{E}(t))$, given by lemma 2.3, is in the interior of the shape in the middle of figure 6.
- $(\chi_\mathcal{E}(s), \eta_\mathcal{E}(s)) - (\chi_\mathcal{E}(t), \eta_\mathcal{E}(t)) = a(s)\mathbf{x} + b(s)\mathbf{y}$ for all $s \in R$ close to t .
- $\mathbf{x} = (1, e^{C(t)} - 1)$ and $\mathbf{y} = (e^{C(t)} - 1, -1)$.
- $a(s) = a_2(s - t)^2 + O((s - t)^3)$ and $b(s) = b_2(s - t)^3 + O((s - t)^4)$.
- $a_2 \neq 0$ and $b_2 \neq 0$ are given in part (b) of the proof of lemma 2.8.
- The edge curve behaves like a cusp of first order in a neighbourhood of $(\chi_\mathcal{E}(t), \eta_\mathcal{E}(t))$.

Case (3): *When $t \in R_{\lambda-\mu}$ is a root of f_t' of multiplicity 2, the edge curve behaves similarly to case (1).*

Case (4): *The set of all $t \in R_{\lambda-\mu}$ for which t is a root of f_t' of multiplicity 3 is a discrete subset of (a, b) . Moreover, in this case, the edge curve behaves similarly to case (2).*

Case (5): *R_0 , the set of all $t \in \mathbb{R} \setminus \text{Supp}(\mu)$ with $C(t) = 0$, is a discrete subset of (a, b) . Moreover, in this case,*

- $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) = (t, 1)$.
- $(\chi_{\mathcal{E}}(s), \eta_{\mathcal{E}}(s)) - (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) = a(s)\mathbf{x} + b(s)\mathbf{y}$ for all $s \in R$ close to t .
- $\mathbf{x} = (1, 0)$ and $\mathbf{y} = (0, -1)$.
- $a(s) = a_1(s - t) + O((s - t)^2)$ and $b(s) = b_1(s - t)^2 + O((s - t)^3)$.
- $a_1 \neq 0$ and $b_1 \neq 0$ are given in part (c) of the proof of lemma 2.8.
- The edge curve behaves like a parabola with negative curvature in a neighbourhood of $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$, with tangent vector \mathbf{x} and normal vector \mathbf{y} .

Case (6): R_1 is a discrete subset of $[a, b]$. Moreover, when $t \in R_1$ is a root of f'_t of multiplicity 1,

- $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$, given by lemma 2.3, satisfy $\chi_{\mathcal{E}}(t) = t$ and $\eta_{\mathcal{E}}(t) \in (0, 1)$.
- $(\chi_{\mathcal{E}}(s), \eta_{\mathcal{E}}(s)) - (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) = a(s)\mathbf{x} + b(s)\mathbf{y}$ for all $s \in R$ close to t .
- $\mathbf{x} = (0, 1)$ and $\mathbf{y} = (1, 0)$.
- $a(s) = a_1(s - t) + O((s - t)^2)$ and $b(s) = b_1(s - t)^2 + O((s - t)^3)$.
- $a_1 \neq 0$ and $b_1 \neq 0$ are given in part (d) of the proof of lemma 2.8.
- The edge curve behaves like a parabola with negative curvature in a neighbourhood of $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$, with tangent vector \mathbf{x} and normal vector \mathbf{y} .

Case (7): R_1 is a discrete subset of $[a, b]$. Moreover, when $t \in R_1$ is a root of f'_t of multiplicity 2,

- $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$, given by lemma 2.3, satisfy $\chi_{\mathcal{E}}(t) = t$ and $\eta_{\mathcal{E}}(t) \in (0, 1)$.
- $(\chi_{\mathcal{E}}(s), \eta_{\mathcal{E}}(s)) - (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) = a(s)\mathbf{x} + b(s)\mathbf{y}$ for all $s \in R$ close to t .
- $\mathbf{x} = (0, 1)$ and $\mathbf{y} = (1, 0)$.
- $a(s) = a_2(s - t)^2 + O((s - t)^3)$ and $b(s) = b_2(s - t)^3 + O((s - t)^4)$.
- $a_2 \neq 0$ and $b_2 \neq 0$ are given in part (e) of the proof of lemma 2.8.
- The edge curve behaves like a cusp of first order in a neighbourhood of $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$.

Case (8): R_2 is a discrete subset of $[a, b]$. Moreover, when $t \in R_2$ is a root of f'_t of multiplicity 1,

- $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$, given by lemma 2.3, satisfy $\chi_{\mathcal{E}}(t) + \eta_{\mathcal{E}}(t) - 1 = t$ and $\eta_{\mathcal{E}}(t) \in (0, 1)$.
- $(\chi_{\mathcal{E}}(s), \eta_{\mathcal{E}}(s)) - (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) = a(s)\mathbf{x} + b(s)\mathbf{y}$ for all $s \in R$ close to t .
- $\mathbf{x} = (1, -1)$ and $\mathbf{y} = (1, 1)$.
- $a(s) = a_1(s - t) + O((s - t)^2)$ and $b(s) = b_1(s - t)^2 + O((s - t)^3)$.
- $a_1 \neq 0$ and $b_1 \neq 0$ are given in part (f) of the proof of lemma 2.8.
- The edge curve behaves like a parabola with negative curvature in a neighbourhood of $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$, with tangent vector \mathbf{x} and normal vector \mathbf{y} .

Case (9): R_2 is a discrete subset of $[a, b]$. Moreover, when $t \in R_2$ is a root of f'_t of multiplicity 2,

- $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$, given by lemma 2.3, satisfy $\chi_{\mathcal{E}}(t) + \eta_{\mathcal{E}}(t) - 1 = t$ and $\eta_{\mathcal{E}}(t) \in (0, 1)$.
- $(\chi_{\mathcal{E}}(s), \eta_{\mathcal{E}}(s)) - (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) = a(s)\mathbf{x} + b(s)\mathbf{y}$ for all $s \in R$ close to t .
- $\mathbf{x} = (1, -1)$ and $\mathbf{y} = (1, 1)$.
- $a(s) = a_2(s - t)^2 + O((s - t)^3)$ and $b(s) = b_2(s - t)^3 + O((s - t)^4)$.
- $a_2 \neq 0$ and $b_2 \neq 0$ are given in part (g) of the proof of lemma 2.8.
- The edge curve behaves like a cusp of first order in a neighbourhood of $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$.

PROOF. Consider (1). First note, part (a) of corollary 3.2 implies that $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ is in the interior of the shape in the middle of figure 6, since, following the notation of this corollary, S_1, S_2, S_3 are all non-empty. Moreover, lemma 2.7 gives the required Taylor expansion, and the Taylor expansion easily implies that the edge curve behaves like a parabola. It remains to show that the curvature, $k(t)$, is negative at $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$, where

$$k(t) := \frac{\chi'_{\mathcal{E}}(t)\eta''_{\mathcal{E}}(t) - \eta'_{\mathcal{E}}(t)\chi''_{\mathcal{E}}(t)}{(\chi'_{\mathcal{E}}(t)^2 + \eta'_{\mathcal{E}}(t)^2)^{\frac{3}{2}}}.$$

Note, denoting $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$, the Taylor expansion gives,

$$\chi'_{\mathcal{E}}(t)\eta''_{\mathcal{E}}(t) - \eta'_{\mathcal{E}}(t)\chi''_{\mathcal{E}}(t) = 2a_1b_1(x_1y_2 - x_2y_1).$$

Substitute the expressions for a_1 and b_1 from part (a) of lemma 2.8, and substitute $\mathbf{x} = (1, e^{C(t)} - 1)$ and $\mathbf{y} = (e^{C(t)} - 1, -1)$ to get,

$$\chi'_{\mathcal{E}}(t)\eta''_{\mathcal{E}}(t) - \eta'_{\mathcal{E}}(t)\chi''_{\mathcal{E}}(t) = \frac{(e^{C(t)} - 1)^2 f_t'''(t)^2}{e^{C(t)} C'(t)^3}.$$

Part (a) of lemma 2.2 gives $e^{C(t)} > 0$ and $C'(t) < 0$. Also $e^{C(t)} - 1 \neq 0$ since $t \in R_{\mu}$, and $f_t'''(t) \neq 0$ since t is a root of f_t' of multiplicity 2. Therefore $k(t) < 0$, as required. Similarly for case (3).

Consider (2). Note, given $t \in R_{\mu}$ for which t is a root of f_t' of multiplicity 3, equation (23) implies that t is a root of the function $w \mapsto C''(w)(e^{C(w)} - 1) - C'(w)^2(e^{C(w)} + 1)$. Also lemma 2.2 implies that this function is well-defined and analytic for $w \in (\mathbb{R} \setminus \text{Supp}(\mu)) \cup (\mathbb{R} \setminus \text{Supp}(\lambda - \mu))$. Thus, since $R_{\mu} \subset (\mathbb{R} \setminus \text{Supp}(\mu)) \cup (\mathbb{R} \setminus \text{Supp}(\lambda - \mu))$, and since the roots of an analytic function are isolated, the set of all $t \in R_{\mu}$ for which t is a root of f_t' of multiplicity 3 is a discrete set. Also, theorem 3.1 implies that this set is contained in (a, b) , since, using the notation of this theorem, roots of multiplicity 3 are only possible in K , and $K \subset (a, b)$. The remainder of the result for case (2) follows using similar methods to those used for case (1). Similarly for case (4).

Consider (5). Part (a) of lemma 2.2 implies that C has a unique analytic extension to $\mathbb{C} \setminus \text{Supp}(\mu)$. Recall that the roots of an analytic function are isolated. It thus follows that R_0 , the set of all $t \in \mathbb{R} \setminus \text{Supp}(\mu)$ with $C(t) = 0$, is a discrete set. Also, $R_0 \subset (a, b)$, since $\{a, b\} \subset \text{Supp}(\mu)$ (see hypothesis 1.1), and since $C(t) > 0$ for all $t > b$ and $C(t) < 0$ for all $t < a$ (see part (a) of lemma 2.2). Finally, since $C(t) = 0$ for all $t \in R_0$, lemma 2.3 gives $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) = (t, 1)$. The remainder of the result for case (5) follows using similar methods to those used for case (1).

Consider (6). The fact that R_1 is a discrete subset of $[a, b]$ follows from remark 1.1 and the definition of R_1 given in equation (11). Also, lemma 2.3 implies that $\chi_{\mathcal{E}}(t) = t$ and $\eta_{\mathcal{E}}(t) < 1$ for all $t \in R_1$. Finally, part (b) of corollary 3.2 implies that $\eta_{\mathcal{E}}(t) > 0$. The remainder of the result for case (6) follows using similar methods to those used for case (1). Similarly for cases (7,8,9). \square

2.5. Examples. In this section we consider some examples of the measure, μ , of hypothesis 1.1, and apply the results of the previous sections to study \mathcal{L} and \mathcal{E} in each

case. We give expressions for the Cauchy transform (obtained from equation (10)), the set $R = R_\mu \cup R_{\lambda-\mu} \cup R_0 \cup R_1 \cup R_2$ (obtained from the definitions of the sets given in parts (1-5) of theorem 2.3, and the edge curve (obtained from lemma 2.3).

REMARK 2.2. *Notation: In this section, \log always denotes principal value, and $\varphi : [a, b] \rightarrow [0, 1]$ denotes the density of μ . Also, whenever we say that a point $(\chi, \eta) \in \mathcal{E}$ corresponds to a root of multiplicity k , we mean that $t := W_{\mathcal{E}}(\chi, \eta)$ is a root of f'_t of multiplicity k (see lemma 2.6). Finally, when describing $\partial\mathcal{L}$, we will often say:*

- $p_1 \rightarrow^i p_2$
- $p_1 \rightarrow^e p_2$

In both cases p_1 and p_2 are distinct points of $\partial\mathcal{L}$. In the first case we mean that section of $\partial\mathcal{L}$ with end-points p_1 and p_2 , traversed clockwise, including the end-points. In the second case we mean that section of $\partial\mathcal{L}$ with end-points p_1 and p_2 , traversed clockwise, excluding the end-points.

2.5.1. Example 1: Take $\varphi(x) = \frac{1}{2}$ for all $x \in [-1, 1]$. Then,

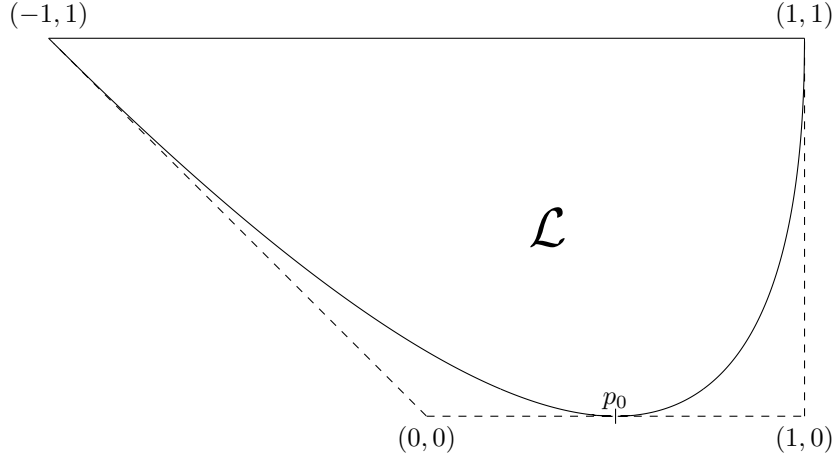
$$\begin{aligned} 2C(w) &= \log(w+1) - \log(w-1) \text{ for all } w \in \mathbb{C} \setminus \mathbb{R}, \\ R &= \mathbb{R} \setminus [-1, 1], \\ R_\mu &= R, \quad R_{\lambda-\mu} = \emptyset, \quad R_0 = \emptyset, \quad R_1 = \emptyset, \quad R_2 = \emptyset, \\ \chi_{\mathcal{E}}(t) &= t - \sqrt{|t+1|} |t-1| (\sqrt{|t+1|} - \sqrt{|t-1|}) \text{ for all } t \in R = \mathbb{R} \setminus [-1, 1], \\ \eta_{\mathcal{E}}(t) &= 1 - \sqrt{|t+1|} |t-1| (\sqrt{|t+1|} - \sqrt{|t-1|})^2 \text{ for all } t \in R = \mathbb{R} \setminus [-1, 1]. \end{aligned}$$

Note that each $t \in [-1, 1]$ satisfies one of the cases of lemma 2.4. Part (2) of lemma 2.5 thus implies that lemmas 2.1, 2.3, and 2.4 give a complete description of $\partial\mathcal{L}$. Plotting this gives figure 9. The dashed lines represent the shape in the middle of figure 6, and the solid lines represent $\partial\mathcal{L}$. In words, following $\partial\mathcal{L}$ in a clockwise direction from the bottom, and using the notation described in remark 2.2:

- $p_0 = (\frac{1}{2}, 0) = (\frac{1}{2} + \int_a^b x\varphi(x)dx, 0)$.
- $p_0 \rightarrow^e (-1, 1)$ is in \mathcal{E}_μ and given by $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ for all $t \in (-\infty, -1) \subset R_\mu$.
- $(-1, 1) \rightarrow^i (1, 1)$ is given by $t \mapsto (t, 1)$ for all $t \in [-1, 1]$.
- $(1, 1) \rightarrow^e p_0$ is in \mathcal{E}_μ and given by $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ for all $t \in (1, +\infty) \subset R_\mu$.

Above, p_0 follows from lemma 2.1, $p_0 \rightarrow^e (-1, 1)$ and $(1, 1) \rightarrow^e p_0$ follow from lemma 2.3, and $(-1, 1) \rightarrow^i (1, 1)$ follows from lemma 2.4.

Note that part (1) of theorem 3.1 implies that each $t \in R = R_\mu = \mathbb{R} \setminus [-1, 1]$ corresponds to a root of multiplicity 2. Case (1) of lemma 2.9 then implies that the edge curve behaves locally like a parabola with negative curvature in a neighbourhood of any edge point. This can clearly be seen in figure 9.

FIGURE 9. \mathcal{L} when $\varphi(x) = \frac{1}{2}$ for all $x \in [-1, 1]$.

2.5.2. **Example 2:** Take $\varphi(x) = \frac{1}{2}$ for all $x \in [0, 1] \cup [2, 3]$. Then,

$$2C(w) = \log(w) - \log(w-1) + \log(w-2) - \log(w-3) \text{ for all } w \in \mathbb{C} \setminus \mathbb{R},$$

$$R = \mathbb{R} \setminus ([0, 1] \cup [2, 3]) = (-\infty, 0) \cup (1, 2) \cup (3, +\infty),$$

$$R_\mu = R \setminus \{3/2\} = (-\infty, 0) \cup (1, 3/2) \cup (3/2, 2) \cup (3, +\infty),$$

$$R_{\lambda-\mu} = \emptyset, \quad R_0 = \{3/2\}, \quad R_1 = \emptyset, \quad R_2 = \emptyset,$$

$$\chi_{\mathcal{E}}(t) = t - 2\sqrt{|t|} |t-1| \sqrt{|t-2|} |t-3| \frac{\sqrt{|t|} |t-2| - \sqrt{|t-1|} |t-3|}{|t-2| |t-3| + |t| |t-1|} \text{ for all } t \in R,$$

$$\eta_{\mathcal{E}}(t) = 1 - 2\sqrt{|t|} |t-1| |t-2| |t-3| \frac{(\sqrt{|t|} |t-2| - \sqrt{|t-1|} |t-3|)^2}{|t-2| |t-3| + |t| |t-1|} \text{ for all } t \in R.$$

As in example 1, lemmas 2.1, 2.3, and 2.4 give a complete description of $\partial\mathcal{L}$. Plotting $\partial\mathcal{L}$ gives figure 10. Following $\partial\mathcal{L}$ in a clockwise direction from the bottom:

- $p_0 = (2, 0) = (\frac{1}{2} + \int_a^b x\varphi(x)dx, 0)$.
- $p_0 \rightarrow^e (0, 1)$ is in \mathcal{E}_μ and given by $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ for all $t \in (-\infty, 0) \subset R_\mu$.
- $(0, 1) \rightarrow^i (1, 1)$ is given by $t \mapsto (t, 1)$ for all $t \in [0, 1]$.
- $(1, 1) \rightarrow^e p_1$ is in \mathcal{E}_μ and given by $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ for all $t \in (1, \frac{3}{2}) \subset R_\mu$.
- $p_1 = (\frac{3}{2}, 1)$ is in \mathcal{E}_0 and given by $p_1 = (\chi_{\mathcal{E}}(\frac{3}{2}), \eta_{\mathcal{E}}(\frac{3}{2}))$ where $\frac{3}{2} \in R_0$.
- $p_1 \rightarrow^e (2, 1)$ is in \mathcal{E}_μ and given by $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ for all $t \in (\frac{3}{2}, 2) \subset R_\mu$.
- $(2, 1) \rightarrow^i (3, 1)$ is given by $t \mapsto (t, 1)$ for all $t \in [2, 3]$.
- $(3, 1) \rightarrow^e p_0$ is in \mathcal{E}_μ and given by $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ for all $t \in (3, +\infty) \subset R_\mu$.

Above, p_0 follows from lemma 2.1. Also, $p_0 \rightarrow^e (0, 1)$, $(1, 1) \rightarrow^e p_1$, $p_1 \rightarrow^e (2, 1)$ and $(3, 1) \rightarrow^e p_0$ follow from lemma 2.3. Finally, $(0, 1) \rightarrow^i (1, 1)$ and $(2, 1) \rightarrow^i (3, 1)$ follow from lemma 2.4.

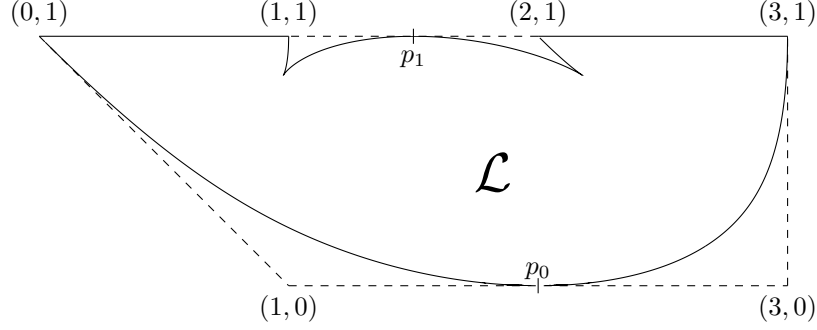


FIGURE 10. \mathcal{L} when $\varphi(x) = \frac{1}{2}$ for all $x \in [0, 1] \cup [2, 3]$.

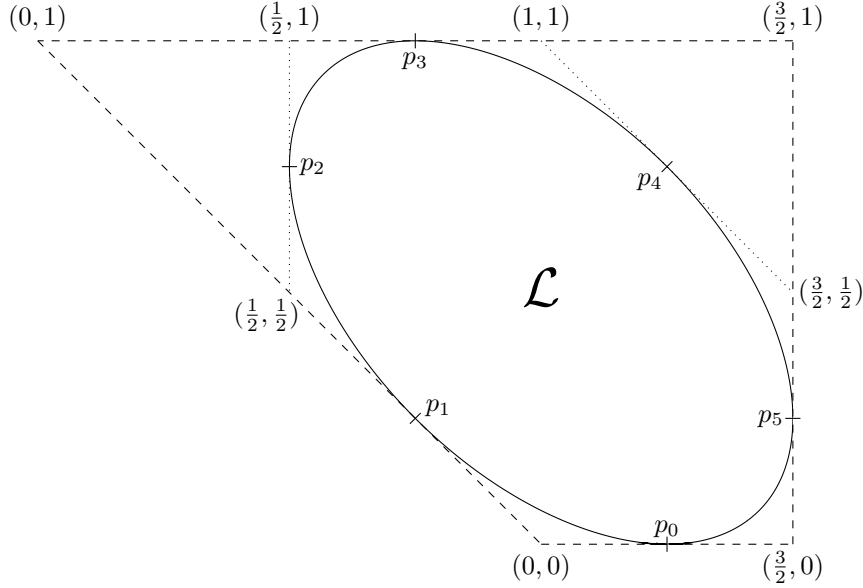
Above we state that $p_1 = (\frac{3}{2}, 1)$ is in \mathcal{E}_0 and given by $p_1 = (\chi_{\mathcal{E}}(\frac{3}{2}), \eta_{\mathcal{E}}(\frac{3}{2}))$ where $\frac{3}{2} \in R_0$. Case (5) of lemma 2.9 thus implies that the edge curve behaves locally like a parabola at $(\frac{3}{2}, 1) = (\chi_{\mathcal{E}}(\frac{3}{2}), \eta_{\mathcal{E}}(\frac{3}{2}))$, and is tangent to the upper boundary at this point. This can clearly be seen in figure 10. We also state that $(1, 1) \rightarrow^e p_1$ is in \mathcal{E}_{μ} . Note from the figure, though we do not attempt to prove this rigorously, that this section contains a cusp. As detailed in cases (1) and (2) of lemma 2.9, the cusp is necessarily of first order and must correspond to a root of multiplicity 3 in R_{μ} , and all other points of $(1, 1) \rightarrow^e p_1$ correspond to roots of multiplicity 2. Similarly for $p_1 \rightarrow^e (2, 1)$. Similar cusps in the edge correspond to a root of multiplicity 3 in $R_{\lambda-\mu}$, as detailed in case (4) of lemma 2.9, but we do not give an example here.

2.5.3. Example 3: Take $\varphi(x) = 1$ for all $x \in [0, \frac{1}{2}] \cup [1, \frac{3}{2}]$. Then,

$$\begin{aligned} C(w) &= \log(w) - \log(w - 1/2) + \log(w - 1) - \log(w - 3/2) \text{ for all } w \in \mathbb{C} \setminus \mathbb{R}, \\ R &= \mathbb{R}, \quad R_{\mu} = (-\infty, 0) \cup (1/2, 3/4) \cup (3/4, 1) \cup (3/2, +\infty), \\ R_{\lambda-\mu} &= (0, 1/2) \cup (1, 3/2), \quad R_0 = \{3/4\}, \quad R_1 = \{1/2, 3/2\}, \quad R_2 = \{0, 1\}, \\ \chi_{\mathcal{E}}(t) &= t - 2(t - 1/2)(t - 3/2) \frac{t(t - 1) - (t - 1/2)(t - 3/2)}{(t - 1)(t - 3/2) + t(t - 1/2)} \text{ for all } t \in R = \mathbb{R}, \\ \eta_{\mathcal{E}}(t) &= 1 - 2 \frac{(t(t - 1) - (t - 1/2)(t - 3/2))^2}{(t - 1)(t - 3/2) + t(t - 1/2)} \text{ for all } t \in R = \mathbb{R}. \end{aligned}$$

Part (1) of lemma 2.5 implies that lemmas 2.1 and 2.3 give a complete description of $\partial\mathcal{L}$. Plotting $\partial\mathcal{L}$ gives figure 11. Following $\partial\mathcal{L}$ in a clockwise direction from the bottom:

- $p_0 = (\frac{5}{4}, 0) = (\frac{1}{2} + \int_a^b x\varphi(x)dx, 0)$.
- $p_0 \rightarrow^e p_1$ is in \mathcal{E}_{μ} and given by $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ for all $t \in (-\infty, 0) \subset R_{\mu}$.
- $p_1 = (\frac{3}{4}, \frac{1}{4})$ is in \mathcal{E}_2 and given by $p_1 = (\chi_{\mathcal{E}}(0), \eta_{\mathcal{E}}(0))$ where $0 \in R_2$.
- $p_1 \rightarrow^e p_2$ is in $\mathcal{E}_{\lambda-\mu}$ and given by $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ for all $t \in (0, \frac{1}{2}) \subset R_{\lambda-\mu}$.
- $p_2 = (\frac{1}{2}, \frac{3}{4})$ is in \mathcal{E}_1 and given by $p_2 = (\chi_{\mathcal{E}}(\frac{1}{2}), \eta_{\mathcal{E}}(\frac{1}{2}))$ where $\frac{1}{2} \in R_1$.
- $p_2 \rightarrow^e p_3$ is in \mathcal{E}_{μ} and given by $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ for all $t \in (\frac{1}{2}, \frac{3}{4}) \subset R_{\mu}$.
- $p_3 = (\frac{3}{4}, 1)$ is in \mathcal{E}_0 and given by $p_3 = (\chi_{\mathcal{E}}(\frac{3}{4}), \eta_{\mathcal{E}}(\frac{3}{4}))$ where $\frac{3}{4} \in R_0$.

FIGURE 11. \mathcal{L} when $\varphi(x) = 1$ for all $x \in [0, \frac{1}{2}] \cup [1, \frac{3}{2}]$.

- $p_3 \rightarrow^e p_4$ is in \mathcal{E}_μ and given by $t \mapsto (\chi_\mathcal{E}(t), \eta_\mathcal{E}(t))$ for all $t \in (\frac{3}{4}, 1) \subset R_\mu$.
- $p_4 = (\frac{5}{4}, \frac{3}{4})$ is in \mathcal{E}_2 and given by $p_4 = (\chi_\mathcal{E}(1), \eta_\mathcal{E}(1))$ where $1 \in R_2$.
- $p_4 \rightarrow^e p_5$ is in $\mathcal{E}_{\lambda-\mu}$ and given by $t \mapsto (\chi_\mathcal{E}(t), \eta_\mathcal{E}(t))$ for all $t \in (1, \frac{3}{2}) \subset R_{\lambda-\mu}$.
- $p_5 = (\frac{3}{2}, \frac{1}{4})$ is in \mathcal{E}_1 and given by $p_5 = (\chi_\mathcal{E}(\frac{3}{2}), \eta_\mathcal{E}(\frac{3}{2}))$ where $\frac{3}{2} \in R_1$.
- $p_5 \rightarrow^e p_0$ is in \mathcal{E}_μ and given by $t \mapsto (\chi_\mathcal{E}(t), \eta_\mathcal{E}(t))$ for all $t \in (\frac{3}{2}, +\infty) \subset R_\mu$.

Above, p_0 follows from lemma 2.1, and the remainder follows from lemma 2.3.

Above we state that $p_5 = (\frac{3}{2}, \frac{1}{4})$ is in \mathcal{E}_1 and given by $p_5 = (\chi_\mathcal{E}(\frac{3}{2}), \eta_\mathcal{E}(\frac{3}{2}))$ where $\frac{3}{2} \in R_1$. It is not difficult to show that $t = \frac{3}{2} \in R_1$ corresponds to a root of multiplicity 1. Thus case (6) of lemma 2.9 implies that the edge curve behaves locally like a parabola at p_5 with tangent vector $(0, 1)$. This can clearly be seen in the figure. Similarly for $p_2 = (\frac{1}{2}, \frac{3}{4}) = (\chi_\mathcal{E}(\frac{1}{2}), \eta_\mathcal{E}(\frac{1}{2})) \in \mathcal{E}_1$, where now we have drawn a dotted line to represent the tangent for clarity. Similarly for $p_4 = (\frac{5}{4}, \frac{3}{4}) = (\chi_\mathcal{E}(1), \eta_\mathcal{E}(1))$ and $p_1 = (\frac{3}{4}, \frac{1}{4}) = (\chi_\mathcal{E}(0), \eta_\mathcal{E}(0))$ which are in \mathcal{E}_2 , except now the tangent vector is $(1, -1)$ (see case (8) of lemma 2.9).

Finally, recall that the asymptotic shape of the frozen boundary of the rescaled regular hexagon is the *Arctic circle*, as shown on the right of figure 2. Recall that we consider tilings of the regular hexagon as tilings of the half-hexagon by adding deterministic lozenges/particles, as shown on the right of figure 3. Also, recall that by shifting these tilings, as described at the beginning of section 1.4, equivalent Gelfand-Tsetlin patterns are obtained. The shifted asymptotic shape of the frozen boundary of the regular hexagon is shown on the right of figure 6. This is identical to figure 11. We therefore have recovered the Arctic circle boundary theorem for the regular hexagon.

2.5.4. Example 4: Take $\varphi(x) := 1$ for all $x \in [0, \frac{1}{3}] \cup [1, \frac{4}{3}] \cup [c, c + \frac{1}{3}]$, where $c := \frac{1}{12}(23 + \sqrt{217}) > \frac{4}{3}$. Define $c_1 := (\frac{1}{2} + \frac{c}{3}) + [(\frac{1}{2} + \frac{c}{3})^2 - \frac{4}{9}(c + \frac{1}{3})]^{1/2}$ and $c_2 := (\frac{1}{2} + \frac{c}{3}) - [(\frac{1}{2} + \frac{c}{3})^2 - \frac{4}{9}(c + \frac{1}{3})]^{1/2}$. Then $c_1 \in (\frac{4}{3}, c)$, $c_2 \in (\frac{1}{3}, 1)$, and,

$$C(w) = \log(w) - \log(w - 1/3) + \log(w - 1) - \log(w - 4/3) \\ + \log(w - c) - \log(w - c - 1/3) \text{ for all } w \in \mathbb{C} \setminus \mathbb{R},$$

$$R = \mathbb{R}, \quad R_\mu = (-\infty, 0) \cup (1/3, c_2) \cup (c_2, 1) \cup (4/3, c_1) \cup (c_1, c) \cup (c + 1/3, +\infty),$$

$$R_{\lambda-\mu} = (0, 1/3) \cup (1, 4/3) \cup (c, c + 1/3),$$

$$R_0 = \{c_1, c_2\}, \quad R_1 = \{1/3, 4/3, c + 1/3\}, \quad R_2 = \{0, 1, c\},$$

$$\chi_{\mathcal{E}}(t) = t - 3(t - 1/3)(t - 4/3)(t - c - 1/3) \frac{A(t)}{B(t)}, \quad \eta_{\mathcal{E}}(t) = 1 - 3 \frac{A(t)^2}{B(t)},$$

$$A(t) := t(t - 1)(t - c) - (t - 1/3)(t - 4/3)(t - c - 1/3),$$

$$B(t) := (t - 1)(t - 4/3)(t - c)(t - c - 1/3) + t(t - 1/3)(t - c)(t - c - 1/3) \\ + t(t - 1/3)(t - 1)(t - 4/3).$$

Part (1) of lemma 2.5 implies that lemmas 2.1 and 2.3 give a complete description of $\partial\mathcal{L}$. Plotting $\partial\mathcal{L}$ gives figure 12. Following $\partial\mathcal{L}$ in a clockwise direction from the bottom:

- $p_0 = (1 + \frac{1}{3}c, 0) = (\frac{1}{2} + \int_a^b x\varphi(x)dx, 0)$.
- $p_0 \rightarrow^e p_1$ is in \mathcal{E}_μ and given by $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ for all $t \in (-\infty, 0) \subset R_\mu$.
- $p_1 = (\frac{4}{9} + \frac{4}{27c}, \frac{5}{9} - \frac{4}{27c})$ is in \mathcal{E}_2 and given by $p_1 = (\chi_{\mathcal{E}}(0), \eta_{\mathcal{E}}(0))$ where $0 \in R_2$.
- $p_1 \rightarrow^e p_2$ is in $\mathcal{E}_{\lambda-\mu}$ and given by $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ for all $t \in (0, \frac{1}{3}) \subset R_{\lambda-\mu}$.
- $p_2 = (\frac{1}{3}, \frac{7}{9} + \frac{2}{27c})$ is in \mathcal{E}_1 and given by $p_2 = (\chi_{\mathcal{E}}(\frac{1}{3}), \eta_{\mathcal{E}}(\frac{1}{3}))$ where $\frac{1}{3} \in R_1$.
- $p_2 \rightarrow^e p_3$ is in \mathcal{E}_μ and given by $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ for all $t \in (\frac{1}{3}, c_2) \subset R_\mu$.
- $p_3 = (c_2, 1)$ is in \mathcal{E}_0 and given by $p_3 = (\chi_{\mathcal{E}}(c_2), \eta_{\mathcal{E}}(c_2))$ where $c_2 \in R_0$.
- $p_3 \rightarrow^e p_4$ is in \mathcal{E}_μ and given by $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ for all $t \in (c_2, 1) \subset R_\mu$.
- $p_4 = (\frac{11}{9} + \frac{2}{27(c-1)}, \frac{7}{9} - \frac{2}{27(c-1)})$ is in \mathcal{E}_2 , given by $p_4 = (\chi_{\mathcal{E}}(1), \eta_{\mathcal{E}}(1))$ where $1 \in R_2$.
- $p_4 \rightarrow^e p_5$ is in $\mathcal{E}_{\lambda-\mu}$ and given by $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ for all $t \in (1, \frac{4}{3}) \subset R_{\lambda-\mu}$.
- $p_5 = (\frac{4}{3}, \frac{5}{9} + \frac{4}{27(c-1)})$ is in \mathcal{E}_1 and given by $p_5 = (\chi_{\mathcal{E}}(\frac{4}{3}), \eta_{\mathcal{E}}(\frac{4}{3}))$ where $\frac{4}{3} \in R_1$.
- $p_5 \rightarrow^e p_6$ is in \mathcal{E}_μ and given by $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ for all $t \in (\frac{4}{3}, c_1) \subset R_\mu$.
- $p_6 = (c_1, 1)$ is in \mathcal{E}_0 and given by $p_6 = (\chi_{\mathcal{E}}(c_1), \eta_{\mathcal{E}}(c_1))$ where $c_1 \in R_0$.
- $p_6 \rightarrow^e p_7$ is in \mathcal{E}_μ and given by $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ for all $t \in (c_1, c) \subset R_\mu$.
- $p_7 = (c + \frac{(c-\frac{1}{3})(c-\frac{4}{3})}{3c(c-1)}, 1 - \frac{(c-\frac{1}{3})(c-\frac{4}{3})}{3c(c-1)})$ is in \mathcal{E}_2 , given by $p_7 = (\chi_{\mathcal{E}}(c), \eta_{\mathcal{E}}(c))$ for $c \in R_2$.
- $p_7 \rightarrow^e p_8$ is in $\mathcal{E}_{\lambda-\mu}$ and given by $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ for all $t \in (c, c + \frac{1}{3}) \subset R_{\lambda-\mu}$.
- $p_8 = (c + \frac{1}{3}, 1 - \frac{(c+\frac{1}{3})(c-\frac{2}{3})}{3c(c-1)})$ is in \mathcal{E}_1 , given by $p_8 = (\chi_{\mathcal{E}}(c + \frac{1}{3}), \eta_{\mathcal{E}}(c + \frac{1}{3}))$ for $c + \frac{1}{3} \in R_1$.
- $p_8 \rightarrow^e p_0$ is in \mathcal{E}_μ and given by $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ for all $t \in (c + \frac{1}{3}, +\infty) \subset R_\mu$.

Above, p_0 follows from lemma 2.1, and the remainder follows from lemma 2.3.

Above we state that $p_5 = (\frac{4}{3}, \frac{5}{9} + \frac{4}{27(c-1)})$ is in \mathcal{E}_1 and given by $p_5 = (\chi_{\mathcal{E}}(\frac{4}{3}), \eta_{\mathcal{E}}(\frac{4}{3}))$ where $\frac{4}{3} \in R_1$. It is not difficult to show that $t = \frac{4}{3} \in R_1$ corresponds to a root of

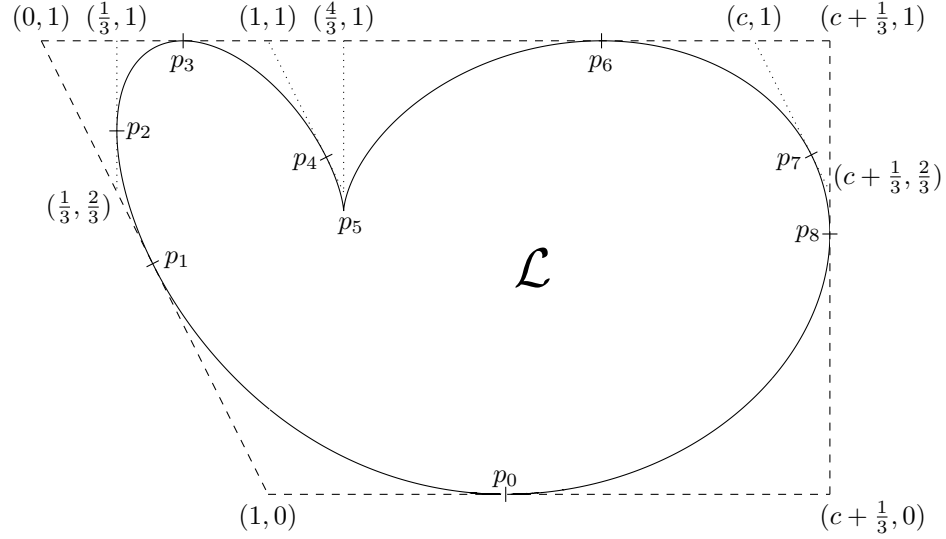


FIGURE 12. \mathcal{L} when $\varphi(x) := 1$ for all $x \in [0, \frac{1}{3}] \cup [1, \frac{4}{3}] \cup [c, c + \frac{1}{3}]$, where $c := \frac{1}{12}(23 + \sqrt{217})$. The vertical direction has been scaled by 2 for clarity.

multiplicity 2. (Here we use the fact that c , as defined above, is a root of $6c^2 - 23c + 13$.) Thus case (7) of lemma 2.9 implies that the edge curve has a cusp of first order at p_5 with orientation $\mathbf{x}(t) = (0, 1)$. This can clearly be seen in the figure. Cusps in the edge with orientation $\mathbf{x}(t) = (1, -1)$ correspond to roots, $t \in R_2$, of multiplicity 2, as detailed in case (9) of lemma 2.9, but we do not give an example here.

2.5.5. Example 5: Take $\varphi(x) := 1 - x$ for all $x \in [0, 1]$, and $\varphi(x) := 1 + x$ for all $x \in [-1, 0]$. Then,

$$C(w) = (w + 1) \log(w + 1) - 2w \log(w) + (w - 1) \log(w - 1) \text{ for all } w \in \mathbb{C} \setminus \mathbb{R},$$

$$R = \mathbb{R} \setminus [-1, 1], \quad R_\mu = R, \quad R_{\lambda-\mu} = \emptyset, \quad R_0 = \emptyset, \quad R_1 = \emptyset, \quad R_2 = \emptyset,$$

$$\chi_{\mathcal{E}}(t) = t + \frac{|t + 1|^{t+1} |t|^{-2t} |t - 1|^{t-1} - 1}{|t + 1|^{t+1} |t|^{-2t} |t - 1|^{t-1} (\log |t + 1| - 2 \log |t| + \log |t - 1|)} \text{ for all } t \in R,$$

$$\eta_{\mathcal{E}}(t) = 1 + \frac{(|t + 1|^{t+1} |t|^{-2t} |t - 1|^{t-1} - 1)^2}{|t + 1|^{t+1} |t|^{-2t} |t - 1|^{t-1} (\log |t + 1| - 2 \log |t| + \log |t - 1|)} \text{ for all } t \in R.$$

Note that each $t \in (-1, 0) \cup (0, 1)$ satisfies case (1) of lemma 2.4, and so $(t, 1) \in \partial \mathcal{L}$ for all such t . However, the points $\{-1, 0, 1\}$ do not satisfy any of the cases of lemma 2.4. Thus part (2) of lemma 2.5 does not apply, and so lemmas 2.1, 2.3, and 2.4 only give a partial description of $\partial \mathcal{L}$.

In order to get a complete description, following the proof of part (2) of lemma 2.5, we must consider all possible limits of sequences of the form $\{(\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n))\}_{n \geq 1}$ as $n \rightarrow \infty$, where $\{w_n\}_{n \geq 1} \subset \mathbb{H}$ satisfies $w_n \rightarrow -1, 0$ or 1 . Denote, as in lemma 2.5, $u_n := \operatorname{Re}(w_n)$, $v_n := \operatorname{Im}(w_n)$, $R_n := \operatorname{Re}(C(w_n))$, and $I_n := -\operatorname{Im}(C(w_n))$. Then, using the above expression for C , it is not difficult to see that $R_n \sim O(1)$. Also, for all n sufficiently

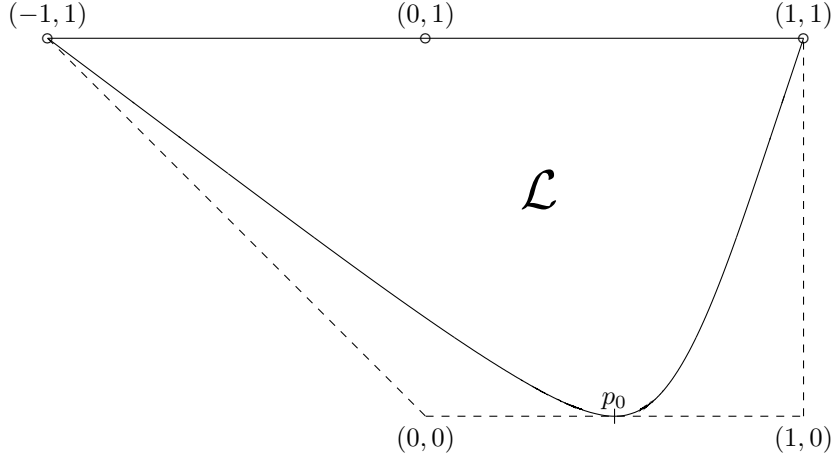


FIGURE 13. \mathcal{L} when $\varphi(x) := 1 - x$ for all $x \in [0, 1]$, and $\varphi(x) := 1 + x$ for all $x \in [-1, 0]$.

large, $\sin(I_n) > -\frac{1}{8}v_n \log((u_n + 1)^2 + v_n^2)$ when $w_n \rightarrow -1$, $\sin(I_n) > -\frac{1}{8}v_n \log(u_n^2 + v_n^2)$ when $w_n \rightarrow 0$, and $\sin(I_n) > -\frac{1}{8}v_n \log((u_n - 1)^2 + v_n^2)$ when $w_n \rightarrow 1$. Equations (13) and (14) finally give $(\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n)) \rightarrow (-1, 1)$ when $w_n \rightarrow -1$, $(\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n)) \rightarrow (0, 1)$ when $w_n \rightarrow 0$, and $(\chi_{\mathcal{L}}(w_n), \eta_{\mathcal{L}}(w_n)) \rightarrow (1, 1)$ when $w_n \rightarrow 1$. These limits, combined with lemmas 2.1, 2.3 and 2.4 give a complete description. Plotting this we get figure 13:

- $p_0 = (\frac{1}{2}, 0) = (\frac{1}{2} + \int_a^b x\varphi(x)dx, 0)$.
- $p_0 \rightarrow^e (-1, 1)$ is in \mathcal{E}_μ and given by $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ for all $t \in (-\infty, -1) \subset R_\mu$.
- $(-1, 1) \rightarrow^i (1, 1)$ is given by $t \mapsto (t, 1)$ for all $t \in [-1, 1]$.
- $(1, 1) \rightarrow^e p_0$ is in \mathcal{E}_μ and given by $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ for all $t \in (1, +\infty) \subset R_\mu$.

Above, p_0 follows from lemma 2.1, and $p_0 \rightarrow^e (-1, 1)$ and $(1, 1) \rightarrow^e p_0$ follow from lemma 2.3. Also, $(-1, 1) \rightarrow^e (0, 1)$ and $(0, 1) \rightarrow^e (1, 1)$ follow from lemma 2.4, and the points $\{(-1, 1), (0, 1), (1, 1)\}$ follow from the direct calculations given above.

2.5.6. Example 6: Take $\varphi(x) = \frac{15}{16}(x - 1)^2(x + 1)^2$ for all $x \in [-1, 1]$. Then,

$$C(w) = \frac{15}{16} \left(\frac{10}{3}w - 2w^3 + (w^2 - 1)^2(\log(w + 1) - \log(w - 1)) \right) \text{ for all } w \in \mathbb{C} \setminus [-1, 1],$$

$$C'(w) = \frac{15}{16} \left(\frac{16}{3} - 8w^2 + 4w(w^2 - 1)(\log(w + 1) - \log(w - 1)) \right) \text{ for all } w \in \mathbb{C} \setminus [-1, 1],$$

$$R = \mathbb{R} \setminus [-1, 1], \quad R_\mu = R, \quad R_{\lambda-\mu} = \emptyset, \quad R_0 = \emptyset, \quad R_1 = \emptyset, \quad R_2 = \emptyset,$$

$$\chi_{\mathcal{E}}(t) = t + \frac{e^{C(t)} - 1}{e^{C(t)}C'(t)} \quad \text{and} \quad \eta_{\mathcal{E}}(t) = 1 + \frac{(e^{C(t)} - 1)^2}{e^{C(t)}C'(t)} \text{ for all } t \in R = \mathbb{R} \setminus [-1, 1].$$

Note that each $t \in (-1, 1)$ satisfies case (1) of lemma 2.4, and so $(t, 1) \in \partial\mathcal{L}$ for all such t . However, the points $\{-1, 1\}$ do not satisfy any of the cases of lemma 2.4. Thus part (2) of lemma 2.5 does not apply, and so lemmas 2.1, 2.3, and 2.4 only give a partial

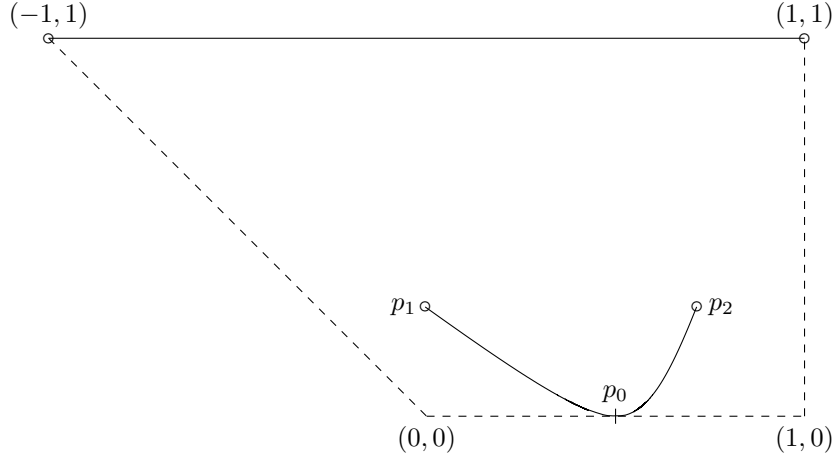


FIGURE 14. Parts of $\partial\mathcal{L}$ when $\varphi(x) = \frac{15}{16}(x-1)^2(x+1)^2$ for all $x \in [-1, 1]$.

description of $\partial\mathcal{L}$. This can clearly be seen in figure 14, where we have plotted those parts of the boundary that we obtain from lemmas 2.1, 2.3, and 2.4:

- $p_0 = (\frac{1}{2}, 0) = (\frac{1}{2} + \int_a^b x\varphi(x)dx, 0)$.
- $p_0 \rightarrow^e p_1$ is in \mathcal{E}_μ and given by $t \mapsto (\chi_\mathcal{E}(t), \eta_\mathcal{E}(t))$ for all $t \in (-\infty, -1) \subset R_\mu$.
- $(-1, 1) \rightarrow^e (1, 1)$ is given by $t \mapsto (t, 1)$ for all $t \in (-1, 1)$.
- $p_2 \rightarrow^e p_0$ is in \mathcal{E}_μ and given by $t \mapsto (\chi_\mathcal{E}(t), \eta_\mathcal{E}(t))$ for all $t \in (1, +\infty) \subset R_\mu$.

Above $p_1 := \lim_{t \uparrow -1} (\chi_\mathcal{E}(t), \eta_\mathcal{E}(t)) \sim (-0.004, 0.290)$ and $p_2 := \lim_{t \downarrow 1} (\chi_\mathcal{E}(t), \eta_\mathcal{E}(t)) \sim (0.714, 0.290)$. Also, p_0 follows from lemma 2.1, $p_0 \rightarrow^e p_1$ and $p_2 \rightarrow^e p_0$ follow from lemma 2.3, and $(-1, 1) \rightarrow^e (1, 1)$ follows from lemma 2.4.

In order to get a complete description, following the proof of part (2) of lemma 2.5, we must consider all possible limits of sequences of the form $\{(\chi_\mathcal{L}(w_n), \eta_\mathcal{L}(w_n))\}_{n \geq 1}$ as $n \rightarrow \infty$, where $\{w_n\}_{n \geq 1} \subset \mathbb{H}$ satisfies $w_n \rightarrow -1$ or 1 . This calculation is beyond the scope of this paper, however, since the technicalities involved in extending lemma 2.4 to cover this situation are highly non-trivial. In the paper [4], we make heavy use of the theory of singular integrals to examine this and other, surprisingly subtle, situations. We obtain a complete description of $\partial\mathcal{L}$ for a broad class of measures.

3. The behaviour of the roots of f'

In this section we examine the behaviour of the roots of the function f' given in equation (7). More generally, it is advantageous to examine the behaviour of the roots of Cauchy transforms of signed-measures of a particular type: Let $A := A_1 \cup \dots \cup A_p$ ($p \geq 1$) and $B := B_1 \cup \dots \cup B_q$ ($q \geq 0$) be unions of disjoint closed intervals. Assume that each A_i and B_j are disjoint, except possibly at their end-points. Let ν^+ and ν^- be non-negative measures with $\text{Supp}(\nu^+) \subset A$ and $\text{Supp}(\nu^-) \subset B$. Finally, assume that the end-points of each A_i are contained in $\text{Supp}(\nu^+)$, and $0 < \nu^+[A_i] < \infty$. Similarly for ν^- and each B_j . Therefore ν^+ is a positive measure, since $p \geq 1$. Also, ν^- is positive when $q \geq 1$, and is 0

when $q = 0$. The Cauchy transform of the signed-measure, $\nu := \nu^+ - \nu^-$, is defined by,

$$(26) \quad g(w) := \int_A \frac{\nu^+[dx]}{w-x} - \int_B \frac{\nu^-[dx]}{w-x},$$

for all $w \in \mathbb{C} \setminus (A \cup B)$. The main result of this section examines the behaviour of the roots of this Cauchy transform:

LEMMA 3.1. *Let $I \subset \mathbb{R} \setminus (A \cup B)$ be an open interval which contains at least $k \geq 0$ roots of g , counting multiplicities. First assume that $\nu^+[A] > \nu^-[B]$. Then, counting multiplicities,*

- (1) g has at most $p+q-1$ roots in $\mathbb{C} \setminus (A \cup B)$.
- (2) g has at most $p+q-2$ roots in I whenever $p+q-1$ is even and either $\{\inf I, \sup I\} \subset A$ or $\{\inf I, \sup I\} \subset B \cup \{\pm\infty\}$.
- (3) g has at most $p+q-2$ roots in I whenever $p+q-1$ is odd and one of $\{\inf I, \sup I\}$ is in A , with the other in $B \cup \{\pm\infty\}$.
- (4) g has at most $p+q-k-2$ roots in $\mathbb{C} \setminus (A \cup B \cup I)$ whenever k is even and either $\{\inf I, \sup I\} \subset A$ or $\{\inf I, \sup I\} \subset B \cup \{\pm\infty\}$.
- (5) g has at most $p+q-k-2$ roots in $\mathbb{C} \setminus (A \cup B \cup I)$ whenever k is odd and one of $\{\inf I, \sup I\}$ is in A , with the other in $B \cup \{\pm\infty\}$.

Next assume that $\nu^+[A] = \nu^-[B]$. Then, counting multiplicities,

- (6) g has at most $p+q-2$ roots in $\mathbb{C} \setminus (A \cup B)$.
- (7) g has at most $p+q-3$ roots in I whenever $p+q-2$ is even and either $\{\inf I, \sup I\} \subset A$ or $\{\inf I, \sup I\} \subset B$.
- (8) g has at most $p+q-3$ roots in I whenever $p+q-2$ is odd and one of $\{\inf I, \sup I\}$ is in A , with the other in B .

PROOF. We will first show:

- (a) The required results hold when ν is of the form,

$$\nu^+ := \sum_{i=1}^M \nu_i^+ \delta_{a_i} \quad \text{and} \quad \nu^- := \sum_{j=1}^N \nu_j^- \delta_{b_j},$$

where $M \geq 1$, $N \geq 0$, $\{a_1, \dots, a_M, b_1, \dots, b_N\} \subset \mathbb{R}$ are distinct, $\nu_i^+ > 0$ and $\nu_j^- > 0$ for all i, j , and each of $\{a_i \in A_1\}, \dots, \{a_i \in A_p\}, \{b_j \in B_1\}, \dots, \{b_j \in B_q\}$ are non-empty.

Next we consider the general case. For all $n \geq 1$, define the measures,

$$\nu_n^+ := \sum_{i=1}^{M_n} \nu_{i,n}^+ \delta_{a_{i,n}} \quad \text{and} \quad \nu_n^- := \sum_{j=1}^{N_n} \nu_{j,n}^- \delta_{b_{j,n}},$$

where $\{a_{1,n}, \dots, a_{M_n,n}\} \subset A$ and $\{b_{1,n}, \dots, b_{N_n,n}\} \subset B$, $\{a_{1,n}, \dots, a_{M_n,n}, b_{1,n}, \dots, b_{N_n,n}\}$ are distinct, and $\nu_{i,n}^+ > 0$ and $\nu_{j,n}^- > 0$ for all i, j . These are constructed such that $\nu_n^+ \rightarrow \nu^+$

and $\nu_n^- \rightarrow \nu^-$ as $n \rightarrow \infty$, in the sense of weak convergence. Also define,

$$(27) \quad g_n(w) := \int_A \frac{\nu_n^+[dx]}{w-x} - \int_B \frac{\nu_n^-[dx]}{w-x},$$

for all $n \geq 1$ and $w \in \mathbb{C} \setminus (A \cup B)$. We will show that:

- (b) Let $w_c \in \mathbb{C} \setminus (A \cup B)$ be a root of g of multiplicity $k \geq 1$, and choose $\epsilon > 0$ such that $B(w_c, 2\epsilon) \subset \mathbb{C} \setminus (A \cup B)$ and w_c is the unique root of g in $B(w_c, 2\epsilon)$. (This is always possible since the roots of an analytic function are isolated.) Then, for all n sufficiently large, g_n has k roots in $B(w_c, \epsilon)$, counting multiplicities.

Finally note, part (a) implies that the required results hold for each g_n . Part (b) then easily gives the required results for g .

Consider (a). Note,

$$g(w) = \sum_{i=1}^M \frac{\nu_i^+}{w-a_i} - \sum_{j=1}^N \frac{\nu_j^-}{w-b_j} = \left(\prod_{k=1}^M \frac{1}{w-a_k} \right) \left(\prod_{l=1}^N \frac{1}{w-b_l} \right) P(w),$$

where P is the polynomial,

$$P(w) = \sum_{i=1}^M \nu_i^+ \left(\prod_{k \neq i} (w-a_k) \right) \left(\prod_l (w-b_l) \right) - \sum_{j=1}^N \nu_j^- \left(\prod_k (w-a_k) \right) \left(\prod_{l \neq j} (w-b_l) \right).$$

Note that $\nu^+[A] = \nu_1^+ + \dots + \nu_M^+$ and $\nu^-[B] = \nu_1^- + \dots + \nu_N^-$. Thus P has degree $M + N - 1$ whenever $\nu^+[A] > \nu^-[B]$, and degree at most $M + N - 2$ whenever $\nu^+[A] = \nu^-[B]$. Also note that, since $\{a_1, \dots, a_M, b_1, \dots, b_N\}$ are distinct, P has no roots in $\{a_1, \dots, a_M, b_1, \dots, b_N\}$. Therefore the roots of P and g coincide, and so g has $M + N - 1$ roots in $\mathbb{C} \setminus \{a_1, \dots, a_M, b_1, \dots, b_N\}$ whenever $\nu^+[A] > \nu^-[B]$, counting multiplicities, and at most $M + N - 2$ roots whenever $\nu^+[A] = \nu^-[B]$.

Alternatively note that we can write,

$$g(w) = \sum_{k=1}^p \sum_{a_i \in A_k} \frac{\nu_i^+}{w-a_i} - \sum_{l=1}^q \sum_{b_j \in B_l} \frac{\nu_j^-}{w-b_j}.$$

Note that each of $\{\{a_i \in A_1\}, \dots, \{a_i \in A_p\}\}$ contains at least 2 points since the end-points of each interval of $A = A_1 \cup \dots \cup A_p$ are contained in $\text{Supp}(\nu^+) = \{a_1, \dots, a_M\}$. Also note that, since the interiors of each A_i and B_j are disjoint and $\{a_1, \dots, a_M, b_1, \dots, b_N\}$ are distinct, there are, for example, no elements from $\{b_1, \dots, b_N\}$ between elements of $\{a_i \in A_1\}$. Also, letting $a_m > a_l$ denote any two neighbouring elements of $\{a_i \in A_1\}$,

$$(28) \quad \lim_{w \in \mathbb{R}, w \uparrow a_m} g(w) = -\infty \quad \text{and} \quad \lim_{w \in \mathbb{R}, w \downarrow a_l} g(w) = +\infty.$$

Thus g has at least 1 root in (a_l, a_m) , counting multiplicities, at least $|\{a_i \in A_1\}| - 1$ roots in $A_1 \setminus \{a_1, \dots, a_M\}$, and at least $M - p$ roots in $A \setminus \{a_1, \dots, a_M\}$ (recall $\{a_1, \dots, a_M\} = \text{Supp}(\nu^+) \subset A = A_1 \cup \dots \cup A_p$). Similarly g has at least $N - q$ roots in $B \setminus \{b_1, \dots, b_N\}$. Thus g has at most $M + N - 1 - (M - p + N - q) = p + q - 1$ roots in $\mathbb{C} \setminus (A \cup B)$.

whenever $\nu^+[A] > \nu^-[B]$, and at most $p + q - 2$ roots whenever $\nu^+[A] = \nu^-[B]$. We have thus shown (1) and (6) for these types of measures.

Now let $I \subset \mathbb{R} \setminus (A \cup B)$ be an open interval which contains at least $k \geq 0$ roots of g , counting multiplicities, with $\{\inf I, \sup I\} \subset A$. First note that $I = (a_l, a_m)$ for some $a_l > a_m$, since the end-points of each interval of $A = A_1 \cup \dots \cup A_p$ are contained in $\text{Supp}(\nu^+) = \{a_1, \dots, a_M\}$. Equation (28) thus implies that g has an odd number of roots in I , counting multiplicities. It thus follows from (1) and (6) that g has at most $p + q - 2$ roots in I whenever $\nu^+[A] > \nu^-[B]$ and $p + q - 1$ is even, and g has at most $p + q - 3$ roots in I whenever $\nu^+[A] = \nu^-[B]$ and $p + q - 2$ is even. Also, whenever k is even and $\nu^+[A] > \nu^-[B]$, (1) implies that g has at most $p + q - 1 - (k + 1) = p + q - k - 2$ roots in $\mathbb{C} \setminus (A \cup B \cup I)$. The other possibilities of (2,3,4,5,7,8) follow in similar way. This gives part (a).

Consider (b). This follows from Rouché's Theorem if we can show that $|g(w)| > |g(w) - g_n(w)|$ for all n sufficiently large and $w \in \partial B(w_c, \epsilon)$, the boundary of $B(w_c, \epsilon)$. We shall show:

$$\inf_{w \in \partial B(w_c, \epsilon)} |g(w)| > 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \sup_{w \in \bar{B}(w_c, \epsilon)} |g(w) - g_n(w)| = 0.$$

The first part follows from the extreme value theorem, since g is analytic in $B(w_c, 2\epsilon)$. Suppose the second part does not hold. Then there exists an $\xi > 0$ for which, for all $n \geq 1$, there exists some $p_n \geq n$ and $z_n \in \bar{B}(w_c, \epsilon)$ with $\xi < |g(z_n) - g_{p_n}(z_n)|$. Choosing $\{z_n\}_{n \geq 1}$ to be convergent, and denoting the limit by z_c ,

$$\xi < |g(z_n) - g(z_c)| + |g(z_c) - g_{p_n}(z_c)| + |g_{p_n}(z_c) - g_{p_n}(z_n)|,$$

for all n . Then, since $B(w_c, 2\epsilon) \subset \mathbb{C} \setminus (A \cup B)$ and $z_n, z_c \in \bar{B}(w_c, \epsilon)$, equation (27) gives

$$\xi < |g(z_n) - g(z_c)| + |g(z_c) - g_{p_n}(z_c)| + \frac{1}{\epsilon^2} (\nu_{p_n}^+[A] + \nu_{p_n}^-[B]) |z_n - z_c|,$$

for all n . Recall that g is analytic in $B(w_c, 2\epsilon)$ and $z_n \rightarrow z_c \in \bar{B}(w_c, \epsilon)$ as $n \rightarrow \infty$. Thus the first term on the right hand side converges to 0 as $n \rightarrow \infty$. Also, recall that $\nu_n^+ \rightarrow \nu^+$ and $\nu_n^- \rightarrow \nu^-$ as $n \rightarrow \infty$, in the sense of weak convergence. Thus the third term on the right hand side converges to 0 as $n \rightarrow \infty$. Finally, weak convergence and equations (26) and (27) imply that the second term on the right hand side converges to 0 as $n \rightarrow \infty$. The inequality thus gives a contradiction, and so the second part must hold, as required. \square

We now consider the behaviour of the roots of f' . Recall (see equation (7)) that $(\chi, \eta) \in [a, b] \times [0, 1]$, $b \geq \chi \geq \chi + \eta - 1 \geq a$, and

$$f'(w) = \int_{\chi}^b \frac{\mu[dx]}{w - x} - \int_{\chi+\eta-1}^{\chi} \frac{(\lambda - \mu)[dx]}{w - x} + \int_a^{\chi+\eta-1} \frac{\mu[dx]}{w - x},$$

for all $w \in \mathbb{C} \setminus S$, where $S := S_1 \cup S_2 \cup S_3$ and

$$(29) \quad S_1 := \text{Supp}(\mu|_{[\chi, b]}), \quad S_2 := \text{Supp}((\lambda - \mu)|_{[\chi+\eta-1, \chi]}), \quad S_3 := \text{Supp}(\mu|_{[a, \chi+\eta-1]}).$$

Note, hypotheses 1.1 and 1.2 imply that the measures are non-negative. Also $\{a, b\} \subset \text{Supp}(\mu)$ and $b - a > 1$. Thus either S_1 or S_3 is non-empty, or both. Also S_2 is non-empty whenever $\eta = 0$. This gives 5 possibilities:

- $\eta > 0$ and S_1, S_2, S_3 are non-empty.
- $\eta = 0$ and S_1, S_2, S_3 are non-empty.
- $\eta > 0$, S_2 is non-empty and either S_1 or S_3 is empty.
- $\eta = 0$, S_2 is non-empty and either S_1 or S_3 is empty.
- $\eta > 0$ and S_2 is empty.

Next write the domain of f' as the disjoint union,

$$(30) \quad \mathbb{C} \setminus S = (\mathbb{C} \setminus \mathbb{R}) \cup J \cup K,$$

where $J := \cup_{j=1}^4 J_j$, $K := \mathbb{R} \setminus (S \cup J)$, and

- $J_1 := (\sup S, +\infty)$.
- $J_2 := (-\infty, \inf S)$.
- $J_3 := (\sup S_2, \inf S_1)$ whenever S_1, S_2 are non-empty and $\inf S_1 > \sup S_2$. Otherwise $J_3 := \emptyset$.
- $J_4 := (\sup S_3, \inf S_2)$ whenever S_2, S_3 are non-empty and $\inf S_2 > \sup S_3$. Otherwise $J_4 := \emptyset$.

Note that $K \subset \mathbb{R}$ is open, and so it can be partitioned as $K = \cup_{k=1}^\infty K_k$, where $\{K_1, K_2, \dots\}$ is a set of pairwise disjoint open intervals. This partition is unique up to order, and is either empty, finite, or countable. An example domain of f' , with the above intervals clearly labelled, is given in figure 15.

The behaviour of the roots of f' for each of the 5 possibilities is then:

THEOREM 3.1. *Assume S_1, S_2, S_3 are non-empty. Then $b > \chi > \chi + \eta - 1 > a$, $J_1 = (b, +\infty)$, $J_2 = (-\infty, a)$, $\chi \in J_3$ whenever $\chi \in \mathbb{C} \setminus S$, and $\chi + \eta - 1 \in J_4$ whenever $\chi + \eta - 1 \in \mathbb{C} \setminus S$. Moreover whenever $\eta > 0$, f' has, counting multiplicities,*

- (1) *at most 2 roots in $(\mathbb{C} \setminus \mathbb{R}) \cup J$.*
- (2) *a root in at most one of $\{\mathbb{C} \setminus \mathbb{R}, J_1, J_2, J_3, J_4\}$.*
- (3) *at most 3 roots in any of $\{K_1, K_2, \dots\}$.*
- (4) *at least 2 roots in at most one of $\{K_1, K_2, \dots\}$.*
- (5) *0 roots in $(\mathbb{C} \setminus \mathbb{R}) \cup J$ whenever f' has at least 2 roots in one of $\{K_1, K_2, \dots\}$.*

Also whenever $\eta = 0$, f' has, counting multiplicities,

- (6) *at most 1 root in $(\mathbb{C} \setminus \mathbb{R}) \cup J_1 \cup J_2$.*
- (7) *0 roots in $J_3 \cup J_4$.*
- (8) *at most 1 root in each of $\{K_1, K_2, \dots\}$.*

Now assume that S_2, S_3 are non-empty and S_1 is empty (similar considerations apply when S_1, S_2 are non-empty and S_3 is empty). Then $b = \chi > \chi + \eta - 1 > a$, $J_1 = (\sup S_2, +\infty)$, $J_2 := (-\infty, a)$, $\chi = b \in J_1$, $J_3 = \emptyset$, and $\chi + \eta - 1 \in J_4$ whenever $\chi + \eta - 1 \in \mathbb{C} \setminus S$. Moreover whenever $\eta > 0$, f' has, counting multiplicities,

- (9) *at most 1 root in $(\mathbb{C} \setminus \mathbb{R}) \cup J$.*
- (10) *at most 1 root in each of $\{K_1, K_2, \dots\}$.*

Also whenever $\eta = 0$, f' has, counting multiplicities,

(11) 0 roots in $(\mathbb{C} \setminus \mathbb{R}) \cup J$.

(12) at most 1 root in each of $\{K_1, K_2, \dots\}$.

Finally, assume that $\eta > 0$ and S_2 empty. Then $J_3 = J_4 = \emptyset$. Also, whenever $\chi \in \mathbb{C} \setminus S$ and $\chi + \eta - 1 \in \mathbb{C} \setminus S$, χ and $\chi + \eta - 1$ are both contained in the same element of $\{J_1, J_2, K_1, K_2, \dots\}$. Also, (11) and (12) hold. The case $\eta = 0$ and S_2 empty never occurs.

PROOF. First suppose that S_1, S_2, S_3 are non-empty. Then, since $\{a, b\} \in \text{Supp}(\mu)$, equation (29) gives $b > \chi > \chi + \eta - 1 > a$. Also equation (30) gives $J_1 = (b, +\infty)$, $J_2 := (-\infty, a)$, $\chi \in J_3$ whenever $\chi \in \mathbb{C} \setminus S$, and $\chi + \eta - 1 \in J_4$ whenever $\chi + \eta - 1 \in \mathbb{C} \setminus S$.

Consider the situation depicted on the top of figure 15. Taking $\nu^+ := \mu|_{[\chi, b]} + \mu|_{[a, \chi + \eta - 1]}$ and $\nu^- := (\lambda - \mu)|_{[\chi + \eta - 1, \chi]}$, f' satisfies the requirements of equation (26) for any choice of $A = \cup_i A_i$ and $B = \cup_j B_j$ shown in figure 15. Finally note that $\nu^+[A] > \nu^-[B]$ whenever $\eta > 0$ and $\nu^+[A] = \nu^-[B]$ whenever $\eta = 0$.

Take $\eta > 0$ and consider (1). Taking the first choice of A and B in figure 15, part (1) of lemma 3.1 implies that f' has at most 2 roots in $\mathbb{C} \setminus (A \cup B) = (\mathbb{C} \setminus \mathbb{R}) \cup J$ (here $p = 2$ and $q = 1$), as required. Consider (3). Taking the second choice of A and B , part (1) of lemma 3.1 implies that f' has at most 3 roots in $\mathbb{C} \setminus (A \cup B) = (\mathbb{C} \setminus \mathbb{R}) \cup J \cup K_1$ (here $p = 3$ and $q = 1$). Thus f' has at most 3 roots in K_1 , and similarly for K_2 , as required.

Consider (2). First note that non-real roots occur in complex conjugate pairs. Thus, whenever f' has a root in $\mathbb{C} \setminus \mathbb{R}$, part (1) implies that f' has exactly 2 roots in $\mathbb{C} \setminus \mathbb{R}$, and 0 roots in J . Alternatively suppose that f' has a root in $J_1 = (b, +\infty)$. Then, taking $I := J_1$ and the first choice of A and B in figure 15, part (5) of lemma 3.1 implies that f' has 0 roots in $\mathbb{C} \setminus (A \cup B \cup I) = (\mathbb{C} \setminus \mathbb{R}) \cup J_2 \cup J_3 \cup J_4$ (here $\inf I = b \in A$, $\sup I = +\infty$, $p = 2, q = 1$ and $k = 1$). The other possibilities of part (2) follow in a similar way.

Consider (4). Suppose that f' has at least 2 roots in K_1 . Taking $I = K_1$ and third choice of A and B , part (4) of lemma 3.1 implies that f' has at most 1 root in $\mathbb{C} \setminus (A \cup B \cup I) = (\mathbb{C} \setminus \mathbb{R}) \cup J \cup K_2$ (here $\{\inf I, \sup I\} \subset A$, $p = 4, q = 1$ and $k = 2$). Thus f' has at most 1 root in K_2 whenever f' has at least 2 roots in K_1 , and vice-versa, as required.

Consider (5). Suppose that f' has at least 2 roots in K_1 . Taking $I = K_1$ and the second choice of A and B , part (4) of lemma 3.1 implies that f' has 0 roots in $\mathbb{C} \setminus (A \cup B \cup I) = (\mathbb{C} \setminus \mathbb{R}) \cup J$ (here $\{\inf I, \sup I\} \subset A$, $p = 3, q = 1$ and $k = 2$). Similarly f' has 0 roots in $(\mathbb{C} \setminus \mathbb{R}) \cup J$ whenever K_2 contains at least 2 roots, as required.

Now take $\eta = 0$, i.e. $\nu^+[A] = \nu^-[B]$, and consider (6). Taking the first choice of A and B , part (6) of lemma 3.1 implies that f' has at most 1 root in $\mathbb{C} \setminus (A \cup B) = (\mathbb{C} \setminus \mathbb{R}) \cup J$ (here $p = 2$ and $q = 1$), as required. Consider (7). Taking $I = J_3$ and the first choice of A and B , part (8) of lemma 3.1 implies that f' has 0 roots in $I = J_3$ (here $\inf I \subset B$, $\sup I \subset A$, $p = 2$ and $q = 1$). Similarly f' has 0 roots in J_4 , as required. Finally, consider (8). Taking $I = K_1$ and the second choice of A and B , part (7) of lemma 3.1 implies that f' has at most 1 root in $I = K_1$ (here $\{\inf I, \sup I\} \subset A$, $p = 3$ and $q = 1$). Similarly for K_2 , as required.

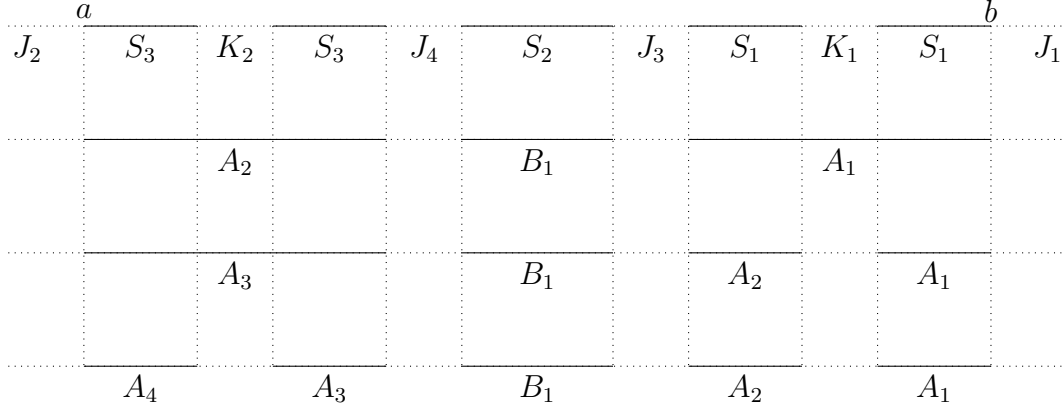


FIGURE 15. A test case of $S = S_1 \cup S_2 \cup S_3$, with the intervals given in equation (30), and various definitions of the sets $A := \cup_i A_i$ and $B := \cup_j B_j$. Solid intervals are closed, and dotted intervals are open.

We have thus shown the required results for the example depicted at the top of figure 15. The result for the general situation when S_1, S_2, S_3 are non-empty follows using similar constructions of $A = \cup_i A_i$ and $B = \cup_j B_j$ to those used above. The other cases of the lemma follow from similar considerations. \square

Recall definitions 1.3 and 1.5 of \mathcal{L} and $\mathcal{E} = \mathcal{E}_\mu \cup \mathcal{E}_{\lambda-\mu} \cup \mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2$:

- \mathcal{L} is the set of all $(\chi, \eta) \in [a, b] \times [0, 1]$ for which $b \geq \chi \geq \chi + \eta - 1 \geq a$ and f' has non-real roots.
- \mathcal{E}_μ is the set of all (χ, η) for which f' has a repeated root in $\mathbb{R} \setminus [\chi + \eta - 1, \chi]$.
- $\mathcal{E}_{\lambda-\mu}$ is the set of all (χ, η) for which f' has a repeated root in $(\chi + \eta - 1, \chi)$.
- \mathcal{E}_0 is the set of all (χ, η) for which $\eta = 1$ and f' has a root at χ ($= \chi + \eta - 1$).
- \mathcal{E}_1 is the set of all (χ, η) for which $\eta < 1$ and f' has a root at χ .
- \mathcal{E}_2 is the set of all (χ, η) for which $\eta < 1$ and f' has a root at $\chi + \eta - 1$.

We end this section by using theorem 3.1 to show the following:

COROLLARY 3.2. *Using the notation of theorem 3.1,*

- (a) S_1, S_2 and S_3 are all non-empty whenever $(\chi, \eta) \in \mathcal{L} \cup \mathcal{E}_\mu \cup \mathcal{E}_{\lambda-\mu}$.
- (b) $\eta > 0$ whenever $(\chi, \eta) \in \mathcal{L} \cup \mathcal{E}$.
- (c) $\{\mathcal{L}, \mathcal{E}_\mu, \mathcal{E}_{\lambda-\mu}, \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2\}$ is pairwise disjoint.

PROOF. Consider (a). First suppose that $(\chi, \eta) \in \mathcal{L}$. Then f' has non-real roots, by definition. Therefore f' has at least 2 roots in $\mathbb{C} \setminus \mathbb{R}$, counting multiplicities, since non-real roots occur in complex conjugate pairs. This is only possible in properties (1-5) of theorem 3.1, and so S_1, S_2, S_3 are non-empty. Next suppose that $(\chi, \eta) \in \mathcal{E}_\mu \cup \mathcal{E}_{\lambda-\mu}$. Then f' has a real-valued repeated root, by definition. Again, this is only possible in properties (1-5), and so S_1, S_2, S_3 are non-empty.

Consider (b). Fix $\eta = 0$. Then either properties (6-8) or (11-12) of theorem 3.1 are satisfied. First suppose that $(\chi, \eta) \in \mathcal{L}$. Recall that f' has at least 2 roots in $\mathbb{C} \setminus \mathbb{R}$,

counting multiplicities. This leads to a contradiction, since it violates either (6) or (11). Next suppose that $(\chi, \eta) \in \mathcal{E}_\mu \cup \mathcal{E}_{\lambda-\mu}$. Then f' has a repeated root, which violates either (6-8) or (11-12). Next suppose that $(\chi, \eta) \in \mathcal{E}_0$. This gives a trivial contradiction, since $\eta = 1$ for all $(\chi, \eta) \in \mathcal{E}_0$.

Finally, suppose that $(\chi, \eta) \in \mathcal{E}_1 \cup \mathcal{E}_2$. Then f' has a root in $\{\chi, \chi + \eta - 1\}$. Recall that properties (6-8) are satisfied whenever S_1, S_2, S_3 are non-empty. In this case $b > \chi > \chi + \eta - 1 > a$, $J_1 = (b, +\infty)$, $J_2 = (-\infty, a)$, $\chi \in J_3$ whenever $\chi \in \mathbb{C} \setminus S$, and $\chi + \eta - 1 \in J_4$ whenever $\chi + \eta - 1 \in \mathbb{C} \setminus S$. Thus f' has a root in $\{\chi, \chi + \eta - 1\} \subset J_3 \cup J_4$, which violates (7). Also recall that properties (11-12) are satisfied when S_2, S_3 are non-empty and S_1 is empty, or alternatively, when S_1, S_2 are non-empty and S_3 is empty. In the first case $b = \chi > \chi + \eta - 1 > a$, $J_1 = (\sup S_2, +\infty)$, $J_2 = (-\infty, a)$, $\chi = b \in J_1$, $J_3 = \emptyset$, and $\chi + \eta - 1 \in J_4$ whenever $\chi + \eta - 1 \in \mathbb{C} \setminus S$. In the second case $b > \chi > \chi + \eta - 1 = a$, $J_1 = (b, +\infty)$, $J_2 = (-\infty, \inf S_2)$, $\chi + \eta - 1 = a \in J_2$, $\chi \in J_3$ whenever $\chi \in \mathbb{C} \setminus S$, and $J_4 = \emptyset$. In either case f' has a root in $\{\chi, \chi + \eta - 1\} \subset J$, which violates (11). (b) then follows by contradiction.

Consider (c). Suppose first that $(\chi, \eta) \in \mathcal{L}$. Recall that f' has at least 2 roots in $\mathbb{C} \setminus \mathbb{R}$, counting multiplicities, and this is only possible when $\eta > 0$ and S_1, S_2, S_3 are non-empty, i.e., properties (1-5) of theorem 3.1. Thus $b > \chi > \chi + \eta - 1 > a$, $J_1 = (b, +\infty)$, $J_2 = (-\infty, a)$, $\chi \in J_3$ whenever $\chi \in \mathbb{C} \setminus S$, and $\chi + \eta - 1 \in J_4$ whenever $\chi + \eta - 1 \in \mathbb{C} \setminus S$. Thus $\eta < 1$, and so $(\chi, \eta) \notin \mathcal{E}_0$. Also, (1) and (5) show that f' has exactly 2 roots in $\mathbb{C} \setminus \mathbb{R}$, counting multiplicities, 0 roots in J , and at most 1 root in each of $\{K_1, K_2, \dots\}$. Thus f' has no real-valued repeated roots, and no roots in $\{\chi, \chi + \eta - 1\} \subset J$, and so $(\chi, \eta) \notin \mathcal{E}_\mu \cup \mathcal{E}_{\lambda-\mu} \cup \mathcal{E}_1 \cup \mathcal{E}_2$.

Next suppose that $(\chi, \eta) \in \mathcal{E}_\mu$, i.e., f' has a repeated root in $\mathbb{R} \setminus [\chi + \eta - 1, \chi]$. There are 2 possibilities (see equation (30)):

- The repeated root is contained in $J \setminus [\chi + \eta - 1, \chi]$.
- The repeated root is contained in $K \setminus [\chi + \eta - 1, \chi]$.

Either case is possible only when $\eta > 0$ and S_1, S_2, S_3 are non-empty, i.e., properties (1-5) of theorem 3.1. Thus $b > \chi > \chi + \eta - 1 > a$, $J_1 = (b, +\infty)$, $J_2 = (-\infty, a)$, $\chi \in J_3$ whenever $\chi \in \mathbb{C} \setminus S$, and $\chi + \eta - 1 \in J_4$ whenever $\chi + \eta - 1 \in \mathbb{C} \setminus S$. Thus $\eta < 1$, and so $(\chi, \eta) \notin \mathcal{E}_0$. Also, (1), (4) and (5) show that f' has at most one real-valued repeated root, and so $(\chi, \eta) \notin \mathcal{E}_{\lambda-\mu}$. Moreover, (5) shows that f' has 0 roots in J whenever the repeated root is contained in $K \setminus [\chi + \eta - 1, \chi]$, and (1) shows that f' has no other roots in J whenever the repeated root is contained in $J \setminus [\chi + \eta - 1, \chi]$. Thus f' has 0 roots in $\{\chi, \chi + \eta - 1\} \subset J$, and so $(\chi, \eta) \notin \mathcal{E}_1 \cup \mathcal{E}_2$.

Next suppose that $(\chi, \eta) \in \mathcal{E}_{\lambda-\mu}$, i.e., f' has a repeated root in $(\chi + \eta - 1, \chi)$. Similar arguments to those used above (in the case $(\chi, \eta) \in \mathcal{E}_\mu$) then give $(\chi, \eta) \notin \mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2$. Next suppose that $(\chi, \eta) \in \mathcal{E}_0$. Then $\eta = 1$, and so $(\chi, \eta) \notin \mathcal{E}_1 \cup \mathcal{E}_2$.

Finally suppose that $(\chi, \eta) \in \mathcal{E}_1 \cap \mathcal{E}_2$, i.e., $\eta < 1$ and f' has a root at χ and at $\chi + \eta - 1$. We show that this contradicts theorem 3.1. First recall, whenever S_1, S_2, S_3 are non-empty, that $b > \chi > \chi + \eta - 1 > a$, $J_1 = (b, +\infty)$, $J_2 = (-\infty, a)$, $\chi \in J_3$ and $\chi + \eta - 1 \in J_4$. Thus f' has a root in J_3 and a root in J_4 , which violates either (2) (when

$\eta > 0$) or (7) (when $\eta = 0$). Also recall, whenever S_2, S_3 are non-empty and S_1 is empty, that $b = \chi > \chi + \eta - 1 > a$, $J_1 = (\sup S_2, +\infty)$, $J_2 = (-\infty, a)$, $\chi = b \in J_1$, $J_3 = \emptyset$, and $\chi + \eta - 1 \in J_4$. Thus f' has a root in J_1 and a root in J_4 , which violates either (9) (when $\eta > 0$) or (11) (when $\eta = 0$). Similarly whenever S_1, S_2 are non-empty and S_3 is empty. Finally recall, whenever $\eta > 0$ and S_2 is empty, that $J_3 = J_4 = \emptyset$, and that χ and $\chi + \eta - 1$ are both contained in the same element of $\{J_1, J_2, K_1, K_2, \dots\}$. Thus $J = J_1 \cup J_2$, and f' has at least 2 roots in an element of $\{J_1, J_2, K_1, K_2, \dots\}$. This violates (11) and (12). Thus we have a contradiction, and so $\mathcal{E}_1 \cap \mathcal{E}_2 = \emptyset$. \square

4. Appendix

4.1. Derivation of the correlation kernel. In section 1.4 we construct a determinantal random point process on configurations of particles in $\mathbb{Z} \times \{1, \dots, n\}$. We now use the Eynard-Mehta theorem (see, for example, proposition 2.13 of Johansson, [10]) to show that this process has the correlation kernel given in equation (1). This is a generalisation of Defosseux, [3], and Metcalfe, [17], which consider a similar process on configurations in $\mathbb{R} \times \{1, \dots, n\}$. The kernel in [3] and [17] is recovered from the kernel in equation (1) using asymptotic arguments. The kernel in equation (1) was also independently obtained by Petrov, [19]. Our proof, based on the methods used in [3] and [17], is more elementary than that of Petrov. In particular, we note that Petrov also uses the Eynard-Mehta theorem. However, he first considers the more complicated $q^{-\text{vol}}$ measure, which is unnecessary. Moreover, we note that the main technical difficulty of the Eynard-Mehta method is to deal with a certain matrix inverse (see equation (34)). We use the finite difference operator to rewrite the expression as a ratio of determinants (see equation (39)), whereas Petrov constructs a diagonal matrix.

We begin by briefly recalling the model. See section 1.4 for more details. Consider all n -tuples, $(y^{(1)}, y^{(2)}, \dots, y^{(n)}) \in \mathbb{Z} \times \mathbb{Z}^2 \times \dots \times \mathbb{Z}^n$, which satisfy

$$y_1^{(r+1)} \geq y_1^{(r)} > y_2^{(r+1)} \geq y_2^{(r)} > \dots \geq y_r^{(r)} > y_{r+1}^{(r+1)},$$

for all r , denoted $y^{(r+1)} \succ y^{(r)}$. Fix $n \geq 1$ and $x \in \mathbb{Z}^n$ with $x_1 > x_2 > \dots > x_n$, and define the following probability measure on the set of all such n -tuples:

$$\nu[(y^{(1)}, \dots, y^{(n)})] := \frac{1}{Z} \cdot \begin{cases} 1 & ; \text{ when } x = y^{(n)} \succ y^{(n-1)} \succ \dots \succ y^{(1)}, \\ 0 & ; \text{ otherwise,} \end{cases}$$

where $Z > 0$ is a normalisation constant.

We now construct a related probability space, the determinantal structure of which is more convenient to examine. Consider all tuples, $(z^{(1)}, \dots, z^{(n-1)}) \in \mathbb{Z}^n \times \dots \times \mathbb{Z}^n$ with

$$z_1^{(r+1)} \geq z_1^{(r)} > z_2^{(r+1)} \geq z_2^{(r)} > \dots > z_n^{(r+1)} \geq z_n^{(r)},$$

for all r , also denoted $z^{(r+1)} \succ z^{(r)}$. Fix $z^{(0)} := (x_n + n - 1, \dots, x_n + 1, x_n)$ and define the following probability measure on the set of all such $(n-1)$ -tuples:

$$(31) \quad \nu'[(z^{(1)}, \dots, z^{(n-1)})] := \frac{1}{Z'} \cdot \begin{cases} 1 & ; \text{ when } x \succ z^{(n-1)} \succ \dots \succ z^{(1)} \succ z^{(0)}, \\ 0 & ; \text{ otherwise,} \end{cases}$$

where $Z' > 0$ is a normalisation constant.

Consider the relationship between the spaces. First note that, whenever $x = y^{(n)} \succ y^{(n-1)} \succ \dots \succ y^{(1)}$ for some $(y^{(1)}, y^{(2)}, \dots, y^{(n)}) \in \mathbb{Z} \times \mathbb{Z}^2 \times \dots \times \mathbb{Z}^n$, then

$$x_1 \geq y_1^{(r)} > \dots > y_r^{(r)} > x_n + n - r - 1,$$

for all $r \leq n$. Whenever $x \succ z^{(n-1)} \succ \dots \succ z^{(1)} \succ z^{(0)}$ for some $(z^{(1)}, \dots, z^{(n-1)}) \in \mathbb{Z}^n \times \mathbb{Z}^n \times \dots \times \mathbb{Z}^n$,

$$(32) \quad \begin{array}{ccccccc} x_1 \geq z_1^{(r)} > \dots > z_r^{(r)} > z_{r+1}^{(r)} & & > z_{r+2}^{(r)} & & > \dots > z_n^{(r)} \\ & & \parallel & & \parallel & & \parallel \\ & & x_n + n - r - 1 & & x_n + n - r - 2 & & \dots & & x_n \end{array}$$

for all $r \leq n - 1$. We refer to $z_1^{(r)}, \dots, z_r^{(r)}$ as the *free particles* of $z^{(r)}$, and $z_{r+1}^{(r)}, \dots, z_n^{(r)}$ as the *deterministic particles*. Note the natural bijection between $\{(y^{(1)}, \dots, y^{(n)}) \in \mathbb{Z} \times \mathbb{Z}^2 \times \dots \times \mathbb{Z}^n : x = y^{(n)} \succ y^{(n-1)} \succ \dots \succ y^{(1)}\}$ and $\{(z^{(1)}, \dots, z^{(n-1)}) \in \mathbb{Z}^n \times \mathbb{Z}^n \times \dots \times \mathbb{Z}^n : x \succ z^{(n-1)} \succ \dots \succ z^{(1)} \succ z^{(0)}\}$: Remove $y^{(n)}$ from each n -tuple $(y^{(1)}, \dots, y^{(n)})$ and map the remaining components, $y^{(r)} = (y_1^{(r)}, \dots, y_r^{(r)})$ for each $r \leq n - 1$, individually as,

$$y^{(r)} \mapsto (y_1^{(r)}, \dots, y_r^{(r)}, x_n + n - r - 1, x_n + n - r - 2, \dots, x_n).$$

The measure ν' is induced by the measure ν under this bijective map. The probabilistic structure of particles in the first space (measure ν) is therefore identical to the probabilistic structure of the free particles in the second space (measure ν'). From now on we restrict to the second space.

A more convenient expression for ν' can be obtained from the work of Warren, [21]:

$$\det \left[1_{z_j^{(r+1)} \geq z_i^{(r)}} \right]_{i,j=1}^n = \begin{cases} 1 & ; \text{ when } z^{(r+1)} \succ z^{(r)}, \\ 0 & ; \text{ otherwise,} \end{cases}$$

for all r . Equation (31) thus gives,

$$(33) \quad \nu'[(z^{(1)}, \dots, z^{(n-1)})] = \frac{1}{Z'} \prod_{r=0}^{n-1} \det \left[\phi_{r,r+1}(z_i^{(r)}, z_j^{(r+1)}) \right]_{i,j=1}^n,$$

where $z^{(n)} := x$, and

$$\phi_{r,r+1}(u, v) := 1_{v \geq u},$$

for all r and $u, v \in \mathbb{Z}$.

Note, each $(z^{(1)}, \dots, z^{(n-1)}) \in \mathbb{Z}^n \times \mathbb{Z}^n \times \dots \times \mathbb{Z}^n$ can be equivalently considered as a configuration of particles in $\mathbb{Z} \times \{1, \dots, n-1\}$ by placing a particle at position $(u, r) \in \mathbb{Z} \times \{1, \dots, n-1\}$ whenever u is an element of $z^{(r)}$. The measure ν' in equation (33) therefore defines a random point process on configurations of particles in $\mathbb{Z} \times \{1, \dots, n-1\}$. Proposition 2.13 of Johansson, [10], proves that this process is determinantal with correlation kernel,

$$(34) \quad K_n((u, r), (v, s)) = \tilde{K}_n((u, r), (v, s)) - \phi_{r,s}(u, v),$$

for all $r, s \in \{1, \dots, n-1\}$ and $u, v \in \mathbb{Z}$, where

$$\tilde{K}_n((u, r), (v, s)) := \sum_{k,l=1}^n \phi_{r,n}(u, z_k^{(n)})(A^{-1})_{kl} \phi_{0,s}(z_l^{(0)}, v),$$

and

$$\begin{aligned} \phi_{r,s}(u, v) &:= 0 \text{ when } s \leq r, \\ \phi_{r,s}(u, v) &:= 1_{v \geq u} \text{ when } s = r+1, \\ \phi_{r,s}(u, v) &:= \sum_{z_1, \dots, z_{s-r-1}} \phi_{r,r+1}(u, z_1) \phi_{r+1,r+2}(z_1, z_2) \cdots \phi_{s-1,s}(z_{s-r-1}, v) \text{ when } s > r+1, \end{aligned}$$

$A \in \mathbb{C}^{n \times n}$ with $A_{kl} := \phi_{0,n}(z_k^{(0)}, z_l^{(n)})$ for all k, l .

Note that, for all $r, s \in \{1, \dots, n\}$ and $u, v \in \mathbb{Z}$,

$$\begin{aligned} \phi_{r,s}(u, v) &= \begin{cases} 0 & ; s \leq r, \\ 1_{v \geq u} & ; s = r+1, \\ \sum_{z_1, \dots, z_{s-r-1}} 1_{v \geq z_{s-r-1} \geq \dots \geq z_2 \geq z_1 \geq u} & ; s > r+1, \end{cases} \\ (35) \quad &= \begin{cases} 0 & ; s \leq r, \\ 1_{v \geq u} & ; s = r+1, \\ 1_{v \geq u} h_{s-r-1}((1)^{v-u+1}) & ; s > r+1, \end{cases} \end{aligned}$$

where $h_r((1)^v)$ denotes the complete homogeneous symmetric polynomial of degree r in v variables, evaluated at the v -tuple $(1)^v := (1, 1, \dots, 1)$. Evaluating gives,

$$(36) \quad \phi_{r,s}(u, v) = \begin{cases} 0 & ; s \leq r, \\ 1_{v-u \geq 0} & ; s = r+1, \\ 1_{v-u \geq 0} \frac{1}{(s-r-1)!} \prod_{j=1}^{s-r-1} (v-u+s-r-j) & ; s > r+1. \end{cases}$$

Thus, noting that $\phi_{r,s}(u, v) = 0$ when $s > r+1$ and $v-u \in \{-1, -2, \dots, -(s-r-1)\}$,

$$(37) \quad \phi_{r,s}(u, v) = 1_{v-u+s-r-1 \geq 0} \begin{cases} 0 & ; s \leq r, \\ 1 & ; s = r+1, \\ \frac{1}{(s-r-1)!} \prod_{j=1}^{s-r-1} (v-u+s-r-j) & ; s > r+1. \end{cases}$$

Finally, letting Δ_v^{n-s} denote $n-s$ iterations of the finite difference operator in the variable v (i.e., $\Delta_v f(v) := f(v+1) - f(v)$ for any function $f : \mathbb{Z} \rightarrow \mathbb{R}$), induction gives

$$(38) \quad \phi_{r,s}(u, v) = 1_{v-u+s-r-1 \geq 0} \Delta_v^{n-s} \frac{1}{(n-r-1)!} \prod_{j=1}^{n-r-1} (v-u+s-r-j),$$

for all $r, s \in \{1, \dots, n\}$ and $u, v \in \mathbb{Z}$.

Recall that a particle is at position (v, s) whenever v is an element of $z^{(s)}$. Thus, since we are only interested in the determinantal structure of the free particles (see equation

(32) and the following comments), it is natural to restrict to $v \geq x_n + n - s$. For all $r, s \in \{1, \dots, n-1\}$, $u \in \mathbb{Z}$ and $v \geq x_n + n - s$, recalling that $z_l^{(0)} = x_n + n - l$,

$$\phi_{0,s}(z_l^{(0)}, v) = \Delta_v^{n-s} \frac{1}{(n-1)!} \prod_{j=1}^{n-1} (v - z_l^{(0)} + s - j) = \Delta_v^{n-s} \phi_{0,n}(z_l^{(0)}, v + s - n),$$

where the first part follows from equation (38), and the second part from equation (37). Equation (34) thus gives,

$$\tilde{K}_n((u, r), (v, s)) = \Delta_v^{n-s} \sum_{k,l=1}^n \phi_{r,n}(u, z_k^{(n)})(A^{-1})_{kl} \phi_{0,n}(z_l^{(0)}, v + s - n).$$

Recall that $z_k^{(n)} = x_k$ and apply Cramer's rule to get,

$$(39) \quad \tilde{K}_n((u, r), (v, s)) = \Delta_v^{n-s} \sum_{k=1}^n \phi_{r,n}(u, x_k) \frac{\det[A(k, v)]}{\det[A]},$$

where $A(k, v) \in \mathbb{C}^{n \times n}$ is the matrix A with column k replaced by

$$(\phi_{0,n}(z_1^{(0)}, v + s - n), \phi_{0,n}(z_2^{(0)}, v + s - n), \dots, \phi_{0,n}(z_n^{(0)}, v + s - n))^T.$$

Note, equation (35) can alternatively be written as,

$$\phi_{r,s}(u, v) = 1_{s>r} h_{v-u}((1)^{s-r}),$$

for all $r, s \in \{1, \dots, n\}$ and $u, v \in \mathbb{Z}$ (by convention $h_k := 0$ whenever $k < 0$). Thus, recalling that $z_i^{(0)} = x_n + n - i$ and $z_j^{(n)} = x_j$, equation (34) gives

$$\det[A] = \det \left[\phi_{0,n}(z_i^{(0)}, z_j^{(n)}) \right]_{i,j=1}^n = \det [h_{x_j - x_n - n + i}((1)^n)]_{i,j=1}^n,$$

Similarly, for all k and v ,

$$\det[A(k, v)] = \det [h_{x(k,v)_j - x_n - n + i}((1)^n)]_{i,j=1}^n,$$

where $x(k, v)_j := x_j$ for all $j \neq k$ and $x(k, v)_k := v + s - n$. Therefore,

$$\frac{\det[A(k, v)]}{\det[A]} = \lim_{q \uparrow 1} \frac{\det[h_{x(k,v)_j - x_n - n + i}(q^{n-1}, \dots, q^1, q^0)]_{i,j}}{\det[h_{x_j - x_n - n + i}(q^{n-1}, \dots, q^1, q^0)]_{i,j}}.$$

The above is written as a limit for computational convenience. Recall that $x_1 > x_2 > \dots > x_n$ and that $v \geq x_n + n - s$. Therefore $x_m - x_n \geq 0$ and $x(k, v)_m - x_n \geq 0$ for all m . Expression (3.7) of MacDonald, [15], thus gives,

$$\frac{\det[A(k, v)]}{\det[A]} = \lim_{q \uparrow 1} \frac{\det[q^{(x(k,v)_j - x_n)(n-i)}]_{i,j}}{\det[q^{(x_j - x_n)(n-i)}]_{i,j}}.$$

The numerator and denominator are both Vandermonde determinants, and so

$$\begin{aligned} \frac{\det[A(k, v)]}{\det[A]} &= \lim_{q \uparrow 1} \prod_{1 \leq i < j \leq n} \left(\frac{q^{x(k, v)_j - x_n} - q^{x(k, v)_i - x_n}}{q^{x_j - x_n} - q^{x_i - x_n}} \right) \\ &= \lim_{q \uparrow 1} \prod_{i=1, i \neq k}^n \left(\frac{q^{v+s-n-x_n} - q^{x_i - x_n}}{q^{x_k - x_n} - q^{x_i - x_n}} \right) = \prod_{i=1, i \neq k}^n \left(\frac{v + s - n - x_i}{x_k - x_i} \right). \end{aligned}$$

Combining with equations (36) and (39) gives,

$$\begin{aligned} \tilde{K}_n((u, r), (v, s)) &= \Delta_v^{n-s} \sum_{k=1}^n \frac{1_{x_k - u \geq 0}}{(n-r-1)!} \prod_{j=1}^{n-r-1} (x_k - u + n - r - j) \prod_{i \neq k} \left(\frac{v + s - n - x_i}{x_k - x_i} \right) \\ &= \Delta_v^{n-s} \sum_{k=1}^n 1_{x_k \geq u} \frac{1}{(n-r-1)!} \prod_{j=u+r-n+1}^{u-1} (x_k - j) \prod_{i \neq k} \left(\frac{v + s - n - x_i}{x_k - x_i} \right), \end{aligned}$$

for all $r, s \in \{1, \dots, n-1\}$, $u \in \mathbb{Z}$ and $v \geq x_n + n - s$. Finally, for any function $f : \mathbb{Z} \rightarrow \mathbb{R}$,

$$(40) \quad (\Delta_v^{n-s} f)(v + s - n) = (n-s)! \sum_{l=v-n+s}^v \frac{f(l)}{\prod_{j=v-n+s, j \neq l}^v (l-j)},$$

and so

$$\tilde{K}_n((u, r), (v, s)) = \frac{(n-s)!}{(n-r-1)!} \sum_{k=1}^n \sum_{l=v-n+s}^v 1_{x_k \geq u} \frac{\prod_{j=u+r-n+1}^{u-1} (x_k - j)}{\prod_{j=v-n+s, j \neq l}^v (l-j)} \prod_{i \neq k} \left(\frac{l - x_i}{x_k - x_i} \right).$$

This, combined with equations (34) and (36) gives the correlation kernel in equation (1), as required. Residue theory gives a natural contour integral representation of this kernel, which is amenable to steepest descent analysis, as shown in equations (4) and (5).

We finish this section by finding an alternate useful expression for $\phi_{r,s}(u, v)$, for all $r, s \in \{1, \dots, n-1\}$ and $u, v \in \mathbb{Z}$. First note that we can replace $1_{v-u+s-r-1 \geq 0}$ in equation (38) by $1_{v \geq u}$. Next note that the product in this equation can be interpreted as a polynomial of degree $n-r-1$ in the variable $v+s-n$. Applying Lagrange interpolation to the polynomial using the n points $x_1 > x_2 > \dots > x_n$ gives,

$$\begin{aligned} \phi_{r,s}(u, v) &= 1_{v \geq u} \Delta_v^{n-s} \frac{1}{(n-r-1)!} \sum_{k=1}^n \prod_{j=1}^{n-r-1} (x_k - u + n - r - j) \prod_{i \neq k} \left(\frac{v + s - n - x_i}{x_k - x_i} \right) \\ &= 1_{v \geq u} \Delta_v^{n-s} \frac{1}{(n-r-1)!} \sum_{k=1}^n \prod_{j=u+r-n+1}^{u-1} (x_k - j) \prod_{i \neq k} \left(\frac{v + s - n - x_i}{x_k - x_i} \right). \end{aligned}$$

Equation (40) finally gives,

$$\phi_{r,s}(u, v) = 1_{v \geq u} \frac{(n-s)!}{(n-r-1)!} \sum_{k=1}^n \sum_{l=v-n+s}^v \frac{\prod_{j=u+r-n+1}^{u-1} (x_k - j)}{\prod_{j=v-n+s, j \neq l}^v (l-j)} \prod_{i \neq k} \left(\frac{l - x_i}{x_k - x_i} \right).$$

4.2. Proof that $W_{\mathcal{L}} : \mathcal{L} \rightarrow \mathbb{H}$ is a diffeomorphism. In theorem 2.1 we defined the function $W_{\mathcal{L}} : \mathcal{L} \rightarrow \mathbb{H}$ and proved that it is a homeomorphism with inverse $(\chi_{\mathcal{L}}(\cdot), \eta_{\mathcal{L}}(\cdot)) : \mathbb{H} \rightarrow \mathcal{L}$. In this section we show the stronger result that $W_{\mathcal{L}}$ is a diffeomorphism. More exactly, we show that $W_{\mathcal{L}} \in C^\infty(\mathcal{L}, \mathbb{H})$ and $(\chi_{\mathcal{L}}(\cdot), \eta_{\mathcal{L}}(\cdot)) \in C^\infty(\mathbb{H}, \mathcal{L})$. Though we do not need it for this paper, we include the proof out of interest.

We begin by briefly recalling the relevant definitions: \mathcal{L} is the set of all $(\chi, \eta) \in [a, b] \times [0, 1]$ for which $b \geq \chi \geq \chi + \eta - 1 \geq a$, and for which

$$(41) \quad f'_{(\chi, \eta)}(w) = \int_a^b \frac{\mu[dx]}{w - x} + \log(w - \chi) - \log(w - \chi - \eta + 1),$$

has non-real roots (here log is principal value). Theorem 3.1 implies that there is a unique root in \mathbb{H} whenever $(\chi, \eta) \in \mathcal{L}$, $W_{\mathcal{L}} : \mathcal{L} \rightarrow \mathbb{H}$ maps to this root, and $(\chi_{\mathcal{L}}(\cdot), \eta_{\mathcal{L}}(\cdot)) : \mathbb{H} \rightarrow \mathcal{L}$ is the inverse of $W_{\mathcal{L}}$.

Define $U := \{(\chi, \eta, u, v) \in \mathbb{R}^4 : (\chi, \eta) \in \mathcal{L}, u \in \mathbb{R}, v > 0\}$, and $\Phi : U \rightarrow \mathbb{R}^2$ by,

$$(42) \quad \Phi(\chi, \eta, u, v) = (\phi_1(\chi, \eta, u, v), \phi_2(\chi, \eta, u, v)) := (\operatorname{Re} f'_{(\chi, \eta)}(u + iv), \operatorname{Im} f'_{(\chi, \eta)}(u + iv)).$$

Note that $\Phi \in C^\infty(U, \mathbb{R}^2)$. Note also that,

$$(43) \quad \begin{aligned} \Phi(\chi, \eta, u, v) = 0 &\Leftrightarrow f'_{(\chi, \eta)}(u + iv) = 0 \\ &\Leftrightarrow W_{\mathcal{L}}(\chi, \eta) = u + iv \\ &\Leftrightarrow (\chi_{\mathcal{L}}(u + iv), \eta_{\mathcal{L}}(u + iv)) = (\chi, \eta). \end{aligned}$$

The Jacobian of Φ with respect to the variables u and v is given by,

$$\frac{\partial(\phi_1, \phi_2)}{\partial(u, v)} = \frac{\partial\phi_1}{\partial u} \frac{\partial\phi_2}{\partial v} - \frac{\partial\phi_1}{\partial v} \frac{\partial\phi_2}{\partial u} = \left(\frac{\partial\phi_1}{\partial u}\right)^2 + \left(\frac{\partial\phi_2}{\partial u}\right)^2 = |f''_{(\chi, \eta)}(u + iv)|^2,$$

for all $(\chi, \eta, u, v) \in U$ (the second step above follows from the Cauchy-Riemann's equations, since $f'_{(\chi, \eta)} : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}$ is analytic). Therefore, whenever $\Phi(\chi_0, \eta_0, u_0, v_0) = 0$ for some fixed $(\chi_0, \eta_0, u_0, v_0) \in U$, equation (43) gives $f'_{(\chi_0, \eta_0)}(u_0 + iv_0) = 0$. Finally, theorem 3.1 implies that $u_0 + iv_0$ is a root of $f'_{(\chi_0, \eta_0)}$ of multiplicity 1, and so

$$\frac{\partial(\phi_1, \phi_2)}{\partial(u, v)}(\chi_0, \eta_0, u_0, v_0) \neq 0.$$

The implicit function theorem (theorem 3.3.1 of [14]) thus implies that there exists functions, f_1 and f_2 , defined on a neighbourhood of (χ_0, η_0) , which are contained in C^∞ (recall, $\Phi \in C^\infty(U, \mathbb{R}^2)$) and which satisfy $\Phi(\chi, \eta, f_1(\chi, \eta), f_2(\chi, \eta)) = 0$ for all (χ, η) in this neighbourhood. Equation (43) thus gives $W_{\mathcal{L}}(\chi, \eta) = f_1(\chi, \eta) + if_2(\chi, \eta)$, for all such (χ, η) , and so $W_{\mathcal{L}} \in C^\infty(\mathcal{L}, \mathbb{H})$.

Also, the Jacobian of Φ with respect to the variables χ and η is given by (see equations (41) and (42)),

$$\frac{\partial(\phi_1, \phi_2)}{\partial(\chi, \eta)} = \frac{\partial\phi_1}{\partial\chi} \frac{\partial\phi_2}{\partial\eta} - \frac{\partial\phi_1}{\partial\eta} \frac{\partial\phi_2}{\partial\chi} = -\frac{(1 - \eta)v}{((u - \chi)^2 + v^2)((u - \chi - \eta + 1)^2 + v^2)} < 0.$$

Thus, fixing $(\chi_0, \eta_0, u_0, v_0) \in U$, the implicit function theorem implies that there exists functions, g_1 and g_2 , defined on a neighbourhood of (u_0, v_0) , which are contained in C^∞ (recall, $\Phi \in C^\infty(U, \mathbb{R}^2)$) and which satisfy $\Phi(g_1(u, v), g_2(u, v), u, v) = 0$ for all (u, v) in this neighbourhood. Equation (43) thus gives $(\chi_{\mathcal{L}}(u + iv), \eta_{\mathcal{L}}(u + iv)) = (g_1(u, v), g_2(u, v))$, for all such (u, v) , and so $(\chi_{\mathcal{L}}(\cdot), \eta_{\mathcal{L}}(\cdot)) \in C^\infty(\mathbb{H}, \mathcal{L})$, as required.

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